9.6. Wednesday for MAT4002

9.6.1. Simplicial Approximation Theorem

Aim: understand homotopy between simplicial complexes $f,g : |K| \to |L|$

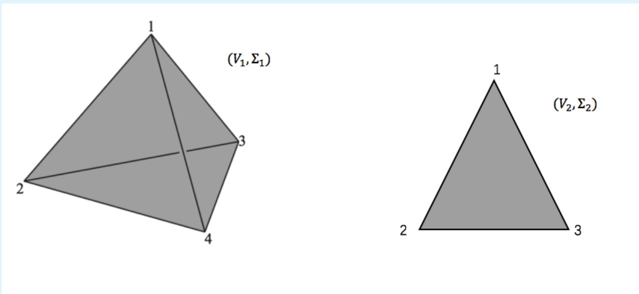
Definition 9.9 [Simplicial Map] A **simplicial map** between $K_1 = (V_1, \Sigma_1)$ and $K_2 = (V_2, \Sigma_2)$ is a mapping $f : K_1 \to K_2$ such that

1. It maps vertexes to vertexes
2. It maps simplicies to simplicies, i.e.,

$$f(\sigma_1) \in \Sigma_2, \ \forall \sigma_1 \in \Sigma_1,$$

■

■ **Example 9.3** For instance, consider the simplicial complexes defined as follows:



In particular, $\{1,2,3,4\} \notin \Sigma_1$ and $\{1,2,3\} \in \Sigma_2$.

In this case, we can define the simplicial map as:

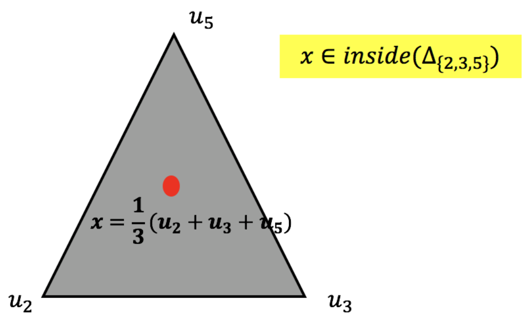
$$f(1) = 1, \quad f(2) = 2, \quad f(3) = 3, \quad f(4) = 3$$

In particular, $f(\{1,2,4\}) = \{1,2,3\} \in \Sigma_2$. ■

Now we want to define the simplicial map between the topological realizations. There are several observations:

Key Observations.

1. We have seen that each $|K| \subseteq \mathbb{R}^m$ for some m. In particular, $m = \#V - 1$.
2. Each point $x \in |K|$ lies uniquely on an inside of some Δ_{σ},, where $\sigma \in \Sigma$.
3. Suppose that the vertices of K_1 are $V_1 = \{u_1, \dots, u_n\} \subseteq \mathbb{R}^m$. Then every $\boldsymbol{x} \in K_1$ can be uniquely written as
$$\boldsymbol{x} = \sum_{i=1}^{k} \alpha_i U_{\sigma_i}$$
with $\alpha_i > 0, \sum \alpha_i = 1$ and $\sigma = \{U_{\sigma_1}, \dots, U_{\sigma_k}\}$ is the unqiue simplex where $x \in \text{inside}(\Delta_\sigma)$.



4. Our simplicial map f maps V_1 to $V_2 = \{w_1, \dots, w_p\} \subseteq \mathbb{R}^m$, so for each i, we have $f(\boldsymbol{u}_i) = \boldsymbol{w}_j$ for some $j \in \{1, \dots, p\}$.

Definition 9.10 [Mapping induced from Simplicial Mapping] The simplicial map $f : K_1 \to K_2$ induces a mapping $|f| : |K_1| \to |K_2|$ between the topological realizations such that

1. It maps vertexes to vertexes, i.e., $|f|(v_1) = f(v_1), \forall v_1 \in V(K_1)$.
2. it is affine, i.e.,
$$|f|\left(\sum_{i=1}^{k} \alpha_i v_i\right) = \sum_{i=1}^{k} \alpha_i |f|(v_i)$$

R $|f| : |K_1| \to |K_2|$ is continuous.

Motivation. Suppose we are given a continuous map $|g| : |K| \to |L|$, we want to approximate $|g|$ by $|f|$, such that $f : K \to L$ is a simplicial map. In this case, f is an easier object to study compared with $|g|$.

We hope to find a mapping f such that $|f| \simeq |g|$. However, we cannot achieve this goal unless we subdivide K into smaller pieces:

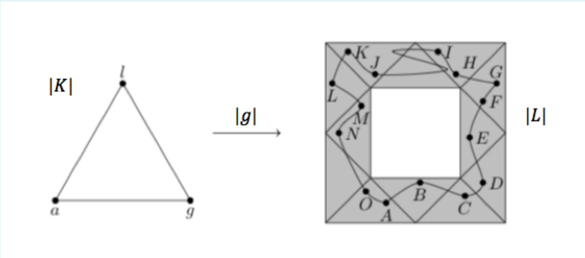
Definition 9.11 [Subdivision] Let K be a simplicial complex. A simplicial complex K' is called a **subdivision** of K if

1. Each simplex of K' is contained in a simplex of K
2. Each simplex of K equals the union of finitely many simplices of K'

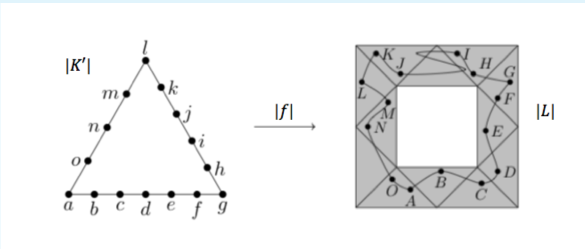
As a result, we can form an homeomorphism $h : |K'| \to |K|$ such that for each $\sigma' \in \Sigma_{K'}$, there exists $\sigma \in \Sigma_K$ satisfying

$$f(\Delta_{\sigma'}) \in \Delta_{\sigma}$$

Example 9.4 Consider the mapping $|g| : |K| \to |L|$ given in the figure below:



Here we denote $|g|(a)$ by A and similarly for the other vertices. It's clear that we can not form a homeomorphism from $|K|$ to $|L|$. One remedy is to subdivide K into smaller pieces as follows:



In this case, it is clear that $|f| : |K'| \to |L|$ is a homeomorphism.

■ **Example 9.5** [Barycentric Subdivision] One typical subdivision is the Barycentric Subdivision:

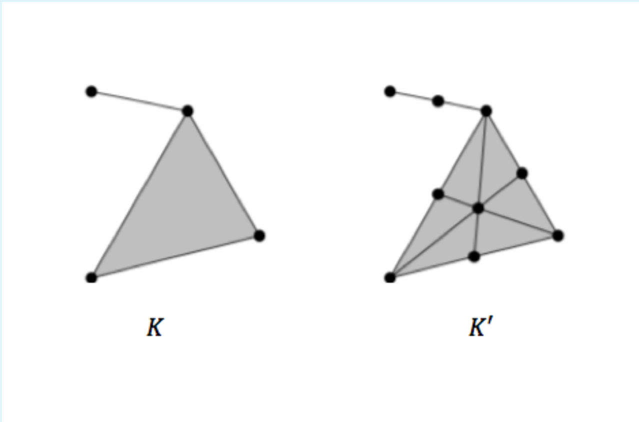


Figure 9.3: Right: the subdivision of K

■

Ⓡ Suppose we have a matric on $|K|$. By subdivision, we can consider $|K'|$ such that for any $\sigma' \in \Sigma_{K'}$, any two points in $\Delta_{\sigma'}$ has a smaller distance.

The following result gives a criterion for the existence of a simplicial approximation for a mapping between topological realizations. For this we recall the notion of star. For a given simplicial complex K, define the star at a vertex v by

$$\text{star}(v) = \bigcup_{v \in \sigma} \sigma^{\circ}.$$

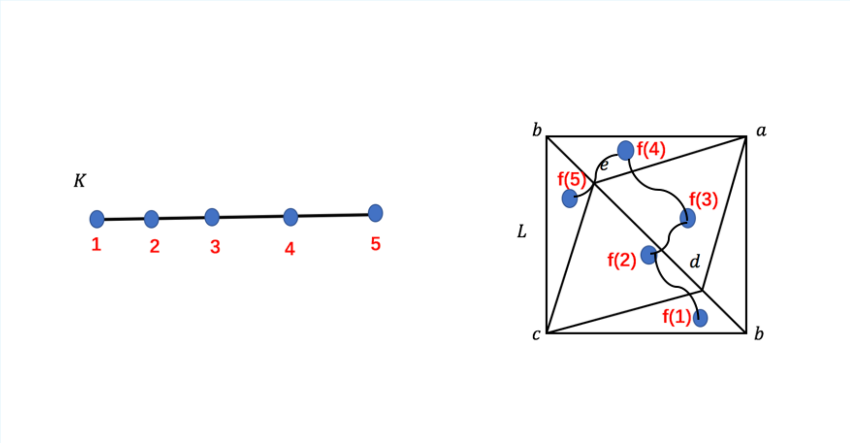
Proposition 9.7 Let $f : |K| \to |L|$ be a continuous mapping. Suppose that for each $v \in V_K$, there exists $g(v) \in V_L$ such that

$$f(\text{st}_K(v)) \subseteq \text{st}_L(g(v)),$$

then the mapping $g : V_K \to V_K$ gives $|g| \simeq f$.

In particular, g is called the **simplicial approximation** to f.

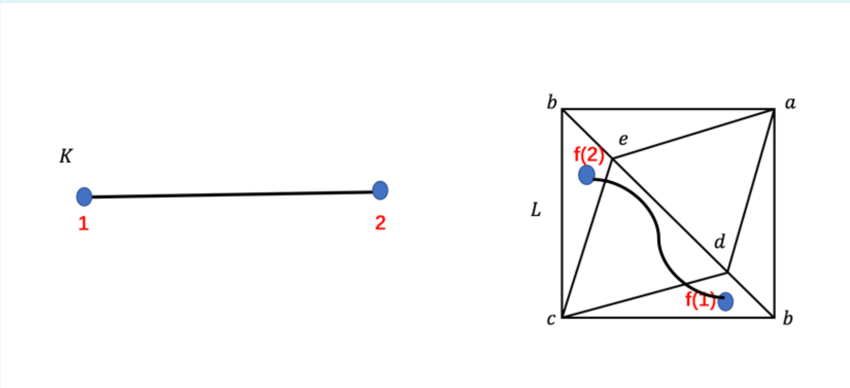
▪ **Example 9.6** 1. First, we give an example of mapping f such that an simplicial approximation exists:



We could define the simplicial approximation g with

$$g(1) = b, g(2) = e, g(3) = e, g(4) = d, g(5) = d \text{ or } c$$

2. In the example below, the simplicial approximation does not exist:



Theorem 9.4 — **Simplicial Approximation.** Let K, L be simplicial complexes with V_K finite, and $f : |K| \to |L|$ be continuous. Then there eixsts a subdivision $|K'|$ of $|K|$ and a simplicial map g such that $|g| \simeq f$.