8.5. Wednesday for MAT4002

Reviewing. We can construct a continuous injection from $|K|$ to $|K'|$, where $K=(V,\Sigma)$ is a simplicial complex, and $K'=(V',\Sigma')$ is its subcomplex:

Let $D_\Sigma := \coprod_{\sigma\in\Sigma}\sigma$ and $D_{\Sigma'} := \coprod_{\sigma'\in\Sigma'}\sigma'$, then $|K'| = D_{\Sigma'}/\sim_{\Sigma'}$ and $|K| = D_\Sigma/\sim_\Sigma$, which follows that

$$f: D_{\Sigma'} \to D_\Sigma \xrightarrow{P} D_\Sigma/\sim_\Sigma, \quad P \text{ denotes the canonical projection mapping}$$

The whole mapping f descends to a continuous mapping

$$\tilde{f}: D_{\Sigma'}/\sim_{\Sigma'} \to D_\Sigma/\sim_\Sigma$$

The $\tilde{f}$ is injective since

$$x\sim_{\Sigma'} y \Longleftrightarrow i(x)\sim_\Sigma i(y), \qquad \forall x,y\in D_\Sigma, \tag{8.6}$$

where i denotes the inclusion mapping.

Another way is to consider the inclusion $i:|K'|\to|K|$, which is continuous and injective as well. Note that $i(|K'|)$ is closed in $|K|$.

Proposition 8.6 For each $K=(V,\Sigma)$, and finite V, there is a continuous injection $g:|K|\hookrightarrow\mathbb{R}^n$ for some n.

Proof. Consider $K^p := (V,\Sigma^p)$, where Σ^p is the power set of V. Therefore, $|K^p| = \Delta^{|V|-1}\subseteq \mathbb{R}^{|V|}$, and K is a simplicial subcomplex of K^p, which follows that

$$l: |K'| \xrightarrow{i} |K^p| \xrightarrow{i} \mathbb{R}^{|V|}$$

The whole mapping l is an inclusion mapping from $|K'|$ to $\mathbb{R}^{|V|}$, which is continuous and injective. The proof is complete. ■

Proposition 8.7 — **Hausdorff.** If $K=(V,\Sigma)$ with fintie V, then $|K|$ is Hausdorff.

Proof. Let $g:|K|\xrightarrow{l}\mathbb{R}^n$. Consider the bijective $g:|K|\to g(|K|)$, which is continuous.

Sicne $|K|$ is compact, and $g(|K|) \subseteq \mathbb{R}^n$ is Hausdorff, we imply that $|K|$ and $g(|K|)$ are homeomorphic, i.e., $|K|$ is Hausdorff. ■

Definition 8.13 [Edge Path] An **edge path** of $K = (V, \Sigma)$ is a sequence of vertices $(v_1, \dots, v_n), v_i \in V$ such that $\{v_i, v_{i+1}\} \in \Sigma, \forall i$.

Proposition 8.8 — **Connectedness.** Let $K = (V, \Sigma)$ be a simplicial complex. TFAE:

1. $|K|$ is connected
2. $|K|$ is path-connected
3. Any 2 vertices in (V, Σ) can be joined by an edge path, i.e., for $\forall u, v \in V$, there exists $v_1, \dots, v_k \in V$ such that $(u, v_1, \dots, v_k, v)$ is an edge path.

Sketch of Proof (to be revised).

1. (3) implies (2): For every $x, y \in |K|$,

$$\begin{cases} x \in \Delta_{\sigma_1} \text{ for some } \sigma_1 \in \Sigma. \\ y \in \Delta_{\sigma_2} \text{ for some } \sigma_2 \in \Sigma. \end{cases}$$

   Take a path joining x to a vertex $v_1 \in \sigma_1$ and a path joining y to a vertex $v_2 \in \sigma_2$. By (3), we have a path joninig v_1 and v_2.

2. (1) implies (3): Suppose on the contrary that there is a vertex v not satisfying (3). Take V' as the set of vertexs that can be joined with v; and V'' as the set of vertexs that cannot be joinied with v.

   Then $V', V'' \neq \emptyset$. Consider K', K'' be simplicial subcomplexes of K, spanned by V' and V''. Then $|K'|, |K''|$ are disjoint, closed in $|K|$.

   $|K| = |K'| \cup |K''|$. If there exists $x \in |K| \setminus (|K'| \cup |K''|)$, then for any $\sigma \in \Sigma$ such that $x \in \Delta_\sigma$, we imply $\Delta_\sigma \not\subseteq |K'|$ or $|K''|$.

   Therefore, σ consists of vertices in both V' and V''. Then there is $v', v'' \in \sigma$ joining V' and V''.

   Therefore, there is no such x and hence $|K| = |K'| \cup |K''|$ is a disjoint union of two closed sets, i.e., not connected.

■

8.5.1. Homotopy

Yoneda's "philosophy". To understand an object X (in our focus, X denotes topological space), we should understand functions

$$f : A \to X, \quad \text{or} \quad g : X \to B$$

One special example is to let $B = \mathbb{R}$.

There are many type of continuous mappings from X to Y. We will group all these mappings into equivalence classes.

Definition 8.14 [Homotopy] A **Homotopy** between two continuous maps $f, g : X \to Y$ is a continuous map

$$H : X \times [0,1] \to Y$$

such that

$$H(x,0) = f(x), \quad H(x,1) = g(x)$$

If such H exists, we say f and g are **homotopic**, denoted as $f \simeq g$.

Example 8.7 Let $Y \subseteq \mathbb{R}^2$ be a convex subset. Consider two continuous maps $f : X \to Y$ and $g : X \to Y$. They are always homotopic since we can define the homotopy

$$H(x,t) = tg(x) + (1-t)f(x)$$

Proposition 8.9 Homotopy is an equivalent relation.

Proof.

1. Let $f : X \to Y$ be any continuous map. Then $f \simeq f$: we can define a homotopy $H(x,t) = f(x), \forall 0 \le t \le 1$.
2. Suppose $f \simeq g$, i.e., H is a homotopy between f and g, then $g \simeq f$: Define the mapping $H'(x,t) = H(x, 1-t)$, then

$$H'(x,0) = g(x), \quad H'(x,1) = f(x)$$

3. Let $f, g, h : X \to Y$ be three continuous maps. If f and g are homotopic and g and h are homotopic, then f and h are homotopic:

   Let $H : X \times [0,1] \to Y$ be a continuous map such that

   $$H(x,0) = f(x), H(x,1) = g(x);$$

   $K : X \times [0,1] \to Y$ be a continuous map such that

   $$K(x,0) = g(x), K(x,1) = h(x).$$

   Define a function $J : X \times [0,1] \to Y$ by

   $$J(x,t) = \begin{cases} H(x,2t), & 0 \le t \le 1/2 \\ K(x,2t-1), & 1/2 \le t \le 1 \end{cases}$$

   - J is continuous, since for all closed $V \subseteq Y$,

     $$J^{-1}(V) = (J^{-1}(V) \cap (X \times [0,1/2])) \cup (J^{-1}(V) \cap (X \times [1/2,1])) = H^{-1}(V) \cup K^{-1}(V),$$

     and the closedness of $H^{-1}(V)$ and $K^{-1}(V)$ implies the closedness of $J^{-1}(V)$
   - Moreover, J has the property that $J(x,0) = H(x,0) = f(x)$, while $J(x,1) = K(x,1) = h(x)$.

$\blacksquare$

R There are only one equivalence class in example (8.7). Actually, for given space X and Y, if any two continuous mapping are homotopic, then we imply there is only one equivalence class.