7.5. Wednesday for MAT4002

7.5.1. Remarks on Triangulation

Consider the simplical complex $K = (V, \Sigma)$ with

$$V = \{1,2,3,4,\ldots,9\}, \quad \Sigma = \begin{cases} \text{9 subsets with 1 element} \\ \text{27 subsets with 2 elements} \\ \text{18 subsets with 3 elements} \end{cases}$$

We start to build the topological realization of K with 9 **0**-simplicies, 27 **1**-simplicies, and 18 **2**-simplicies. The identification of them is as follows:

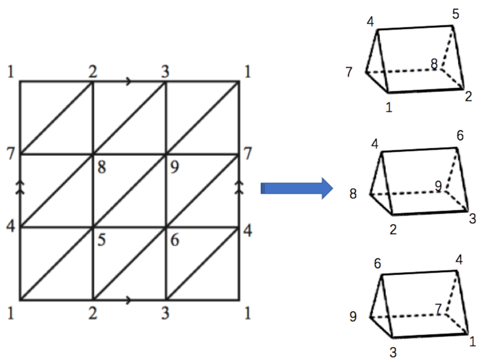


Figure 7.1: Step 1: Identify 3 columns separately, i.e., identify $\{1,7,4,1,2,8,5,2\}$, $\{2,8,5,2,3,9,6,3\}$, and $\{3,9,6,3,1,7,4,1\}$.

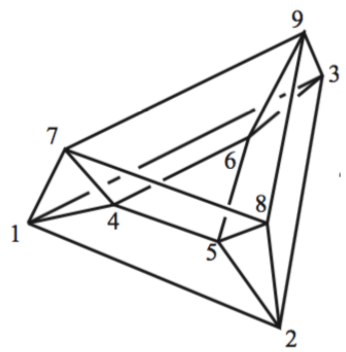
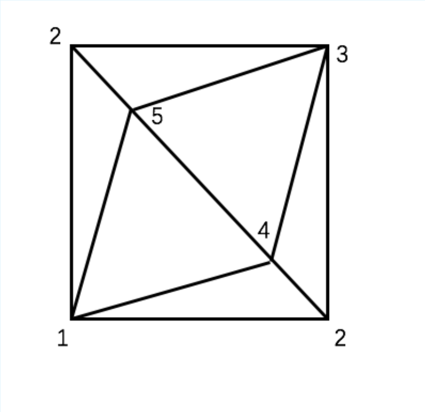


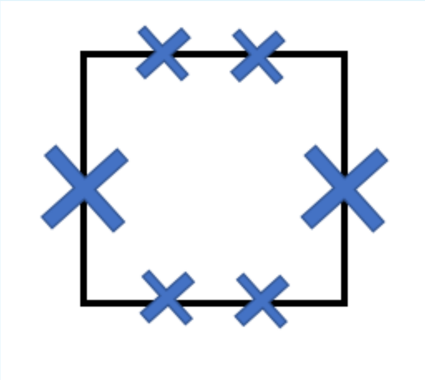
Figure 7.2: Step 2: "gluing" these three prisms in the figure above together.

Question: why K is homeomorphic to the torus?

■ **Example 7.9** Consider the simplicial complex (V, Σ) described below:

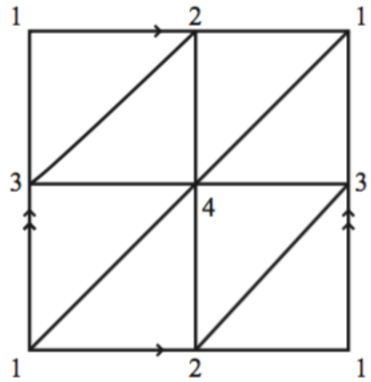


The $|(V, \Sigma)|$ is homeomorphism to the quotient space S^1 plotted below

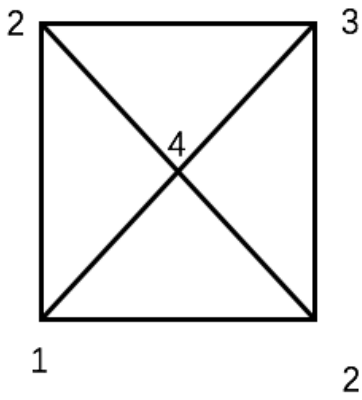


■

Furthermore, can we build a triangulation of the tours using fewer simplices? The answer is no. Consider the figure below: at the bottom edge of this square, there are two 1-simplicies lablled $\{1,2\}$, which cannot happen in a tours.



Interesting question: does the triangulation of the figure below leads to S^2? Answer: No.



The simplicial complex gives us another way to study X, i.e., it suffices to study (V,Σ) such that $|(V,\Sigma)| \cong X$. The question is that can we distinguish $X = S^1 \times S^1$ and $Y = S^2$? In other words, can we distinguish the difference of corresponding topological realizations?

Theorem 7.2 — Euler's Formula. Suppose that $|(V_1,\Sigma_1)| \cong |(V_2,\Sigma_2)|$, then

$$\begin{aligned}&\sum_{i=1}^{\infty}(-1)^i \text{(number of subsets in } \Sigma_1 \text{ with } (i+1)\text{-element)}\\ =&\sum_{i=1}^{\infty}(-1)^i \text{(number of subsets in } \Sigma_2 \text{ with } (i+1)\text{-element)}\end{aligned}$$

From previous examples we can see that $\mathcal{X}(S^2) = 5 - 9 + 6 = 2$ and $\mathcal{X}(S^1 \times S^1) =$

$9-27+18=0$, which implies

$$S^2 \not\cong S^1 \times S^1.$$

7.5.2. Simplicial Subcomplex

Definition 7.12 [Simplicial Subcomplex] A subcomplex of a simplicial complex $K=(V,\Sigma)$ is a simplicial complex $K'=(V',\Sigma')$ such that

$$V' \subseteq V, \quad \Sigma' \subseteq \Sigma$$

Proposition 7.14 Suppose K' is subcomplex of K, then $|K'|$ is closed in $|K|$.

Proof. Suppose that D is the disjoint union of all the simplicial complex forming $|K|$. (note that the number of component in D is $|\Sigma|$)

Consider the canonical projection mapping $D \to |K|$. Observe that $p^{-1}(|K'|)$ precisely equals to $\coprod_{\sigma' \in \Sigma'} \sigma'$, which is closed in D. By definition of quotient topology, $|K'|$ is also closed. ∎

Definition 7.13 [Subcomplex spanned by vertices] Let $K=(V,\Sigma)$ be a simplicial complex and $V' \subseteq V$. Then the subcomplex spanned by V' is (V',Σ') such that

- V' denotes the vertex set.
- the simplices Σ' is given by
$$\{\sigma \in \Sigma \mid \sigma \subseteq V'\}$$

Definition 7.14 [Link and Star] Let $(V,\Sigma)=K$ be simplicial complex

- The **link** of $v \in V$, denoted as $\mathrm{lk}(v)$ is the sub-complex with

– vertex set

$$\{\boldsymbol{w} \in V \setminus \{\boldsymbol{v}\} \mid \{\boldsymbol{v}, \boldsymbol{w}\} \in \Sigma\}$$

– simplicies

$$\{\sigma \in \Sigma \mid \boldsymbol{v} \notin \sigma \text{ and } \sigma \cup \{\boldsymbol{v}\} \in \Sigma\}$$

- The star of $\boldsymbol{v}$ (denoted as $\mathrm{st}(\boldsymbol{v})$) is

$$\bigcup \{\mathrm{inside}(\sigma) \mid \sigma \in \Sigma, \boldsymbol{v} \in \sigma\}$$

Proposition 7.15 $\mathrm{st}(\boldsymbol{v})$ is open and $\boldsymbol{v} \in \mathrm{st}(\boldsymbol{v})$.

Proof. Omitted. ■

In fact, $|K| \setminus \mathrm{st}(\boldsymbol{v})$ is the simplicial subcomplex spanned by V.

7.5.3. Some properties of simplicial complex

Proposition 7.16 Suppose that $K = (V, \Sigma)$, where V is finite. Then $|K|$ is compact.

Proof. The mapping $p : D \to |K|$ is a canonical projection mapping, which is continuous; and D (the finite disjoint union of Δ_σ's) is compact.

Therefore, $p(D) = |K|$ is compact. ■

Proposition 7.17 For any simplicial complex $K = (V, \Sigma)$, where V is finite, there is a continuous injection

$$f : |K| \to \mathbb{R}^n \quad \text{for some } n$$

Proof. Let $K' = (V, \Sigma')$, where $\Sigma' =$ power set of V. Then

$$|K'| = \Delta^{|V|-1} \subseteq \mathbb{R}^{|V|}$$

Consider the inclusion

$$i : |K| \to |K'|$$

which comes from the following:

1. Consider the $D := \coprod_{\sigma\in\Sigma} \Delta_\sigma$ and $D' = \coprod_{\sigma'\in\Sigma'} \Delta_{\sigma'}$ in (V,Σ) and (V,Σ')
2. Construct the mapping $\tilde{i} : D \hookrightarrow D' \xrightarrow{p'} |K|$.
3. The mapping $\tilde{i}$ descends to $i : D/\sim \longrightarrow |K'|$ (try to write down the detailed mapping), which is continuous and injective.

Therefore, $|K| \hookrightarrow |K'|$, i.e., $|K| \hookrightarrow \mathbb{R}^n$. The proof is complete.

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