6.3. Monday for MAT4002

6.3.1. Quotient Topology

Now given a topologcal space X and an equivalence relation $\sim$ on it, our goal is to construct a topology on the space $X/\sim$.

Proposition 6.1 Suppose $(X,\mathcal{T})$ is a topological space, and $\sim$ is an equivalene relation on X. Define the canonical projection map:

$$\begin{array}{ll} p: & X \to X/\sim \\ \text{with} & x \to [x] \end{array},$$

which assigns each point $x \in X$ into the equivalence class $[x]$. Then define a family of subsets $\tilde{\mathcal{T}}$ on $X/\sim$ by:

$$\tilde{U} \subseteq X/\sim \text{ is in } \tilde{\mathcal{T}} \text{ if } p^{-1}(\tilde{U}) \text{ is in } \mathcal{T}$$

Then $\tilde{\mathcal{T}}$ is a topology for $X/\sim$, called the **quotient topology**, and $(X/\sim,\tilde{\mathcal{T}})$ is called the quotient space, and $p: X \to X/\sim$ is called the **natural map**.

Proof.

1. $p^{-1}(X/\sim) = X \in \mathcal{T}$ and $p^{-1}(\emptyset) = \emptyset \in \mathcal{T}$, which implies $X/\sim \in \tilde{\mathcal{T}}$ and $\emptyset \in \tilde{\mathcal{T}}$.
2. Suppose that $\tilde{U},\tilde{V} \in \tilde{\mathcal{T}}$, then we imply

$$p^{-1}(\tilde{U}),p^{-1}(\tilde{V}) \in \mathcal{T} \implies p^{-1}(\tilde{U}\cap\tilde{V}) \in \mathcal{T},$$

   i.e., $\tilde{U}\cap\tilde{V} \in \tilde{\mathcal{T}}$.
3. Following the similar argument in (2), and the relation

$$p^{-1}\left(\bigcup \tilde{U}_i\right) = \bigcup p^{-1}(\tilde{U}_i),$$

   we conclude that $\tilde{T}$ is closed under countably union.

The proof is complete. ■

R

1. The proposition (6.1) claims that $\tilde{U}$ is open in $X/\sim$ iff $p^{-1}(\tilde{U})$ is open in X. The general question is that, does $p(U)$ is open in $X/\sim$, given that U is open in X? This may not necessarily hold. (See example (6.4)) In general $p^{-1}(p(U))$ is strictly larger than U, and may not be necessarily open in X, even when U is open.
2. By definition, we can show that p is continuous.

To fill the gap on the question shown in the remark, we consider the notion of the open mapping:

Definition 6.3 [Open Mapping] A function $f: X \to Y$ between two topological spaces is an **open mapping** if for each open U in X, $f(U)$ is open in Y. ■

R From the remark above, we can see that:

1. Not every continuous mapping is an open mapping
2. The canonical projection mapping p is not necessarily be an open mapping.

■ **Example 6.4** 1. The mapping $p: [0,1] \times [0,1] \to ([0,1] \times [0,1])/\sim$ sending the square to the Mobius band M is not an open mapping:

Consider the open ball $U = B_{1/2}((0,0))$ in $[0,1] \times [0,1]$. Note that $p(U)$ is open in M iff $p^{-1}(p(U))$ is open in $[0,1] \times [0,1]$. We can calculate $p^{-1}(p(U))$ explicitly:

$$p^{-1}(p(U)) = U \cup \{(1,y) \mid 1/2 \leq y \leq 1\},$$

which is not open.

■

6.3.2. Properties in quotient spaces

6.3.2.1. Closedness on $X/\sim$

Proposition 6.2 A subset $\tilde{V}$ is closed in the quotient space $X/\sim$ iff $p^1(\tilde{V})$ is closed in X, where $p : X \to X/\sim$ denotes the canonical projection mapping.

Proof. It follows from the fact that

$$p^{-1}\left((X/\sim) \setminus \tilde{V}\right) = X \setminus p^{-1}(\tilde{V})$$

■

6.3.2.2. Isomorphism on $X/\sim$

The quotient space can be used to study other type of spaces:

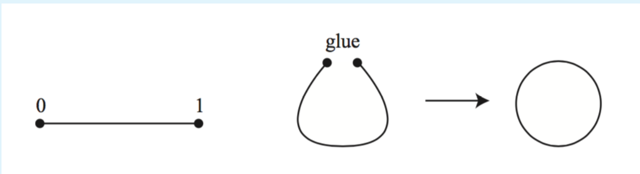
■ **Example 6.5** Consider $X = [0,1]$. We define $x_1 \sim x_2$ if:

$$x_1 = 0, x_2 = 1, \quad \text{or} \quad x_1 = 1, x_2 = 0$$

In other words, the partition on X is given by:

$$X = \{0,1\} \cup \big(\bigcup_{x \in (0,1)} \{x\} \big)$$

The quotient space seems "glue" the endpoints of the interval $[0,1]$ together, shown in the figure below:



It is intuitive that the constructed quotient space should be homeomorphic to a circle S^1. We will give a formal proof on this fact. ■

Proposition 6.3 Let X and Z be topological spaces, and $\sim$ an equivalence relation on X. Let $g : X/\sim \to Z$ be a function, and $p : X \to X/\sim$ is a projection mapping The mapping g is continuous if and only if $g \circ p : X \to Z$ is continuous.

Proof.

1. *Necessity*. Suppose that g is continuous. It's clear that p is continuos, i.e, $g \circ p : X \to Z$ is continuous.
2. *Sufficiency*. Suppose that $g \circ p : X \to Z$ is continuous. Given any open U in Z, we imply $(g \circ p)^{-1}(U) = p^{-1}g^{-1}(U)$ is open in X. By definition of the quotient topology, we imply $g^{-1}(U)$ is open in $X/\sim$. Therefore, g is continuous.

■

(R) This useful lemma can be generalized into the case for generlized canonical projection mapping, called quotient mapping.

> **Definition 6.4** [Quotient mapping] A map $p : X \to Y$ between topological spaces is a **quotient mapping** if
>
> 1. p is **surjective**; and
> 2. p is continuous;
> 3. For any $U \subseteq Y$ such that $p^{-1}(U)$ is open in X, we imply U is open in Y.

The canonical projection map is clearly a quotient map. Actually, a stronger version of proposition (6.3) follows:

Proposition 6.4 Suppose that $p : X \to Y$ is a quotient map and that $g : Y \to Z$ is any mapping to another space Z. Then g is continuous iff $g \circ p$ is continuous.

Proof. The proof follows similarly as in proposition (6.3). ■

Now we give a formal proof of the conclusion in the example (6.5):

Proof. Define the mapping

$$\begin{aligned} f: & \quad [0,1] \to S^1 \\ \text{with} & \quad t \mapsto (\cos 2\pi t, \sin 2\pi t). \end{aligned}$$

Since $f(0) = f(1)$, the function f induces a well-defined function

$$\begin{aligned} g: & \quad [0,1]/\sim \to S^1 \\ \text{with} & \quad [t] \mapsto f(t) \end{aligned}$$

such that $f = g \circ p$, where p denotes the canonical projection mapping. Note that f is continuous. By proposition (6.3), we imply g is continuous. Furthermore,

1. Since $[0,1]$ is compact and p is continuous, we imply $p([0,1]) = [0,1]/\sim$ is compact
2. S^1 is Hausdorff
3. g is a bijection

By applying theorem(5.3), we conclude that g is a homeomorphism, i.e., $[0,1]/\sim$ and S^1 are homeomorphic.

■

The argument in the proof can be generalized into the proposition below:

Proposition 6.5 Let $f: X \to Y$ be a surjective continuous mapping between topologcial spaces. Let $\sim$ be the equivalence relation on X defined by the partition $\{f^{-1}(y) \mid y \in Y\}$ (i.e., $f(x) = (x')$ iff $x \sim x'$). If X is compact and Y is Hausdorff, then $X/\sim$ and Y are homeomorphic.

R The proposition (6.5) is a pattern of argument we should use several times. In order to show $X/\sim$ and Y are homeomorphic, we should think up a surjective continuous mapping $f: X \to Y$ "with respect to the identifications", i.e., $f(x_1) = f(x_2)$ whenever $x_1 \sim x_2$. Therefore f will induce a well-defined function $g: X/\sim \to Y$ such that $f = g \circ f$. Then checking the conditions in theorem(5.3) leads to the desired results.

Torus. We now study the torus in more detail.

1. Consider $X = [0,1] \times [0,1]$ and define $(s_1, t_1) \sim (s_2, t_2)$ if one of the following holds:

   - $s_1 = s_2$ and $t_1 = t_2$;
   - $\{s_1, s_2\} = \{0,1\}$, $t_1 = t_2$;
   - $\{t_1, t_2\} = \{0,1\}$ and $s_1 = s_2$;
   - $\{s_1, s_2\} = \{0,1\}$, $\{t_1, t_2\} = \{0,1\}$

   The corresponding quotient space $([0,1] \times [0,1]) / \sim$ is hoemomorhpic to the 2-dimension torus $\mathbb{T}^2$.

   Proof. Define the mapping $f : [0,1] \times [0,1] \to \mathbb{T}^2$ as $(t_1, t_2) \mapsto (e^{2\pi i t_1}, e^{2\pi i t_2})$.

   (a) f is surjective, which also implies $\mathbb{T}^2 = f([0,1] \times [0,1])$ is compact.

   (b) $\mathbb{T}^2$ is Hausdorff

   (c) It's clear that $(s_1, t_1) \sim (s_2, t_2)$ implies $f(s_1, t_1) = f(s_2, t_2)$. Conversely, suppose

   $$e^{2\pi i s_1} = e^{2\pi i s_2}, \quad e^{2\pi i t_1} = e^{2\pi i t_2}$$

   By the familiar property of e^{ix}, we imply either $t_1 = t_2$ or $\{t_1, t_2\} = \{0,1\}$; and either $s_1 = s_2$ or $\{s_1, s_2\} = \{0,1\}$

   By applying proposition (6.5), we conclude that $([0,1] \times [0,1]) / \sim$ is homeomorphic to $\mathbb{T}^2$. ∎

2. Consider the closed disk $\mathbb{D}^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$, and defube $(x_1, y_1) \sim (x_2, y_2)$ if one of the following holds:

   - $x_1 = x_2$ and $y_1 = y_2$;
   - (x_1, y_1) and (x_2, y_2) are in the boundary circle $\mathbb{S}^1$

   The corresponding quotient space $\mathbb{D}^2 / \sim$ is hoemomorhpic to the 2-dimension sphere $\mathbb{S}^2 = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1\}$.

Proof. Define the mapping

$$\begin{aligned} f: \quad & \mathbb{D}^2 \to \mathbb{S}^2 \\ \text{with} \quad & (0,0) \mapsto (0,0,1) \\ & (x,y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}} \sin(\pi\sqrt{x^2+y^2}), \frac{y}{\sqrt{x^2+y^2}} \sin(\pi\sqrt{x^2+y^2}), \cos(\pi\sqrt{x^2+y^2}) \right) \end{aligned}$$

It's easy to check the conditions in proposition (6.5), and we conclude that $\mathbb{D}^2/\sim$ is hoemomorhpic to $\mathbb{S}^2$ ∎