5.3. Monday for MAT4002

5.3.1. Continuous Functions on Compact Space

Proposition 5.3 Let $f : X \to Y$ be continuous function on topological spaces, with $A \subseteq X$ compact. Then $f(A) \subseteq Y$ is compact.

Proof. Let $\{U_i \mid i \in I\}$ be an open cover of $f(A)$, i.e.,

$$f(A) \subseteq \bigcup_{i \in I} U_i, \quad U_i \in \mathcal{T}_Y$$

It follows that $\{f^{-1}(U_i) \mid i \in I\}$ is an open cover of A:

$$A \subseteq f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i)$$

By the compactness of A, there exists finite subcover of A:

$$A \subseteq \bigcup_{k=1}^{n} f^{-1}(U_{i_k}),$$

which implies the constructed finite subcover of $f(A)$:

$$\begin{aligned} f(A) &\subseteq f(\cup_{k=1}^{n} f^{-1}(U_{i_k})) \\ &= \bigcup_{k=1}^{n} U_{i_k} \end{aligned}$$

∎

Corollary 5.2
1. Suppose that X is compact, and the mapping $f : X \to \mathbb{R}$ is continuous, then $f(X)$ is closed and bounded, i.e., there exists $m, M \in X$ such that $f(m) \leq f(x) \leq f(M)$, $\forall x \in X$.
2. Suppose moreover that X is connected, then

$$f(X) = [f(m), f(M)].$$

Theorem 5.2 The space X, Y are compact iff $X \times Y$ is compact under product topology.

Proof. 1. *Sufficiency:* Given that $X \times Y$ is compact, consider the projection mapping (which is continuous):

$$\begin{cases} P_X : X \times Y \to X \\ P_Y : X \times Y \to Y \end{cases}$$

By applying proposition (5.3), $P_X(X \times Y) = X, P_Y(X \times Y) = Y$ are both compact.

2. *Necessity:* Suppose that $\{W_i\}_{i \in I}$ is an open cover of $X \times Y$. Each open set W_i can be written as:

$$W_i = \bigcup_{j \in \mathcal{J}_i} U_{ij} \times V_{ij}, \quad U_{ij} \in \mathcal{T}_X, V_{ij} \in \mathcal{T}_Y.$$

It follows that

$$X \times Y = \bigcup_{(i,j) \in K} U_{ij} \times V_{ij}, \quad K = \{(i,j) \mid i \in I, j \in \mathcal{J}_i\}$$

Therefore, it suffices to show $\{U_{ij} \times V_{ij} \mid (i,j) \in K\}$ has a finite subcover of $X \times Y$.

- Note that $X \times \{y\} \subseteq \bigcup_{(i,j) \in K} U_{ij} \times V_{ij}$ is compact for each $y \in Y$, which implies there exists finite $S_y \in K$ such that

$$X \times \{y\} \subseteq \bigcup_{s \in S_y} U_s \times V_s$$

- w.l.o.g., assume that $y \in V_s, \forall s \in S_y$, since we can remove the $U_s \times V_s$ such that $y \notin V_s$. Define the set $V_y := \cap_{s \in S_y} V_s$, which is an open set containing y. We imply $\{V_y\}_{y \in Y}$ forms an open cover of Y. By the compactness of Y,

$$\{V_{y_1}, \ldots, V_{y_m}\}$$

forms a finite subcover of Y.

- For each $\ell = 1, \dots, m$,

$$X \times \{y_\ell\} \subseteq \bigcup_{s \in S_{y_\ell}} U_s \times V_s$$

Note that for any $(x,y) \in X \times Y$, there exists $\ell \in \{1, \dots, m\}$ such that $y \in V_{y\ell}$, i.e., $y \in V_s$ for $\forall s \in S_{y\ell}$. Therefore,

$$X \times Y = \bigcup_{\ell=1}^{m} \bigcup_{s \in S_{y_\ell}} U_s \times V_s$$

Now pick

$$I' = \{i \in I \mid (i,j) \in \cup_{\ell=1}^{m} S_{y_\ell}\},$$

we imply $X \times Y = \bigcup_{i' \in I'} W_i$ and I' is finite.

■

Theorem 5.3 Suppose that X is compact, Y is Hausdorff, $f : X \to Y$ is continuous, bijective, then f is a **homeomorphism**.

Proof. It suffices to show f^{-1} is continuous. Therefore, it suffices to show $(f^{-1})^{-1}(V)$ is closed, given that V is closed in X:

Let $V \subseteq X$ be closed. Then V is compact, which implies $f(V)$ is compact. Since $f(V) \subseteq Y$ is Hausdorff, we imply $f(V)$ is compact, i.e., $f(V)$ is closed. ■