4.6. Wednesday for MAT4002

There will be a quiz on Monday.

Reviewing.

- Connectedness / Path-Connectedness

4.6.1. Remark on Connectedness

Proposition 4.14 All path connected spaces X are connected.

Proof. Fix any $x \in X$, for all $y \in X$, there exists a continuous mapping $p_y : [0,1] \to X$ such that

$$p_y(0) = x, \quad p_y(1) = y.$$

Consider $C_y = p_y([0,1])$, which is connected, due to proposition (4.9).

Note that $\{C_y\}_{y \in X}$ is a collection of connected sets, and for any $y, y' \in X$, $C_y \cap C_{y'} \ni \{x\}$ is non-empty. Applying proposition (4.10), we imply $X = \cup_{y \in X} C_y$ is connected. ■

Example 4.5 1. Exercise: if $A \subset B \subset \overline{A}$, then A is connected implies B is connected. (Hint: $U \cap A = \emptyset$ implies $U \cap \overline{A} = \emptyset$.)

Proof. Suppose B is not connected, i.e., for any open U, V such that $B \subseteq U \cup V$ and $(U \cap V) \cap B = \emptyset$, we imply $U \cap B \neq \emptyset$ and $V \cap B \neq \emptyset$, and therefore

$$U \cap \overline{A} \neq \emptyset, \quad V \cap \overline{A} \neq \emptyset$$

which implies

$$U \cap A \neq \emptyset, \quad V \cap A \neq \emptyset$$

which contradicts to the connectedness of A. ■

2. The converse of proposition (4.14) may not be necessarily true. Consider the so-called **Topologist's bomb** example:

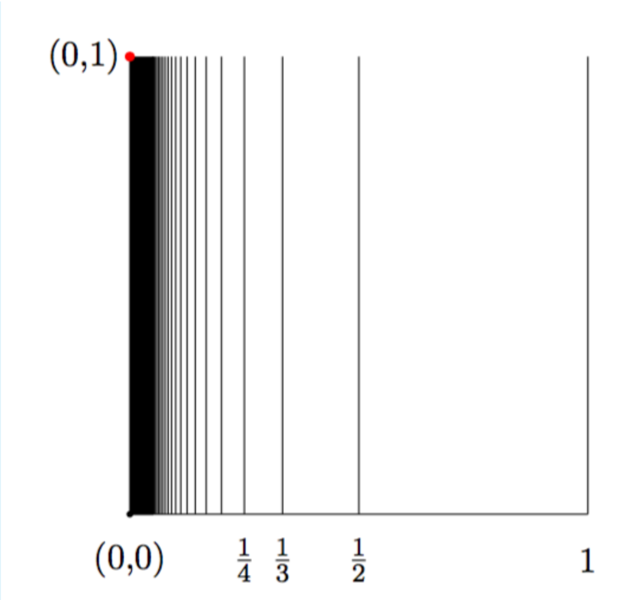

Figure 4.1: Connected space X but not path-connected

Here we construct a connected space $X \subseteq \mathbb{R}^2$ but not path-connected shown in Fig (4.1), i.e., the union of the interval $[0,1]$ together with vertical line segments from $(1/n,0)$ to $(1/n,1)$ and the single point $(0,1)$.

$$X = ([0,1] \times \{0\}) \cup \bigcup_{n \geq 1} (\{1/n\} \times [0,1]) \cup (0,1).$$

(a) Firstly, X is not path-connected. We show that there is no path in X links $(0,1)$ to any other point, i.e., for continuous mapping $p : [0,1] \to X$ with $p(0) = (0,1)$, we may imply $p(t) = (0,1)$ for any t.

Define

$$A = \{t \in [0,1] \mid p(t) = (0,1)\}.$$

We claim that $A = [0,1]$, i.e., suffices to show A is both open and closed in $[0,1]$:

i. The set $A = p^{-1}(\{0,1\})$ is nonempty and closed, since the pre-image of a closed set is closed as well.

ii. The set A is open: choose $t_0 \in A$. By continuity of p, there exists $\delta > 0$ such that

$$\|p(t) - (0,1)\| = \|p(t) - p(t_0)\| < \frac{1}{2}, \quad t \in [0,1] \cap (t_0 - \delta, t_0 + \delta).$$

Since there is no point on the x-axis with the distance $1/2$ to the point $(0,1)$, we imply $p(t)$ is not on the x-axis when $t \in [0,1] \cap (t_0 - \delta, t_0 + \delta)$. Therefore, the x-coordinate of $p(t)$ is either 0 or of the form $1/n$.

It suffices to show the open interval $I := [0,1] \cap (t_0 - \delta, t_0 + \delta)$ is in A. Define the composite function $f = x \circ p : I \to \mathbb{R}$, where the mapping $x : \mathbb{R}^2 \to \mathbb{R}$ is defined as $(a,b) \mapsto a$. Note that I is connected, we imply $f(I)$ is connected, and $f(I)$ belongs to $\{0\} \cup \{1/n\}$.

The only nonempty connected subset of $\{0\} \cup \{1/n\}$ is a single point (left as exercise), and therefore $f(I)$ is a single point. Since $f(t_0) = 0$,we imply $f(I) = \{0\}$, i.e., $I \subseteq A$. Therefore A is open.

■

4.6.2. Completeness

Compact set in X is used to generalize “closed and bounded” in $\mathbb{R}^n$.

Definition 4.11 Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{U} = \{U_i \mid i \in I\}$ of open sets is an open cover of X if

$$X = \bigcup_{i \in I} U_i$$

A subcover of $\mathcal{U}$ is a subfamily

$$\mathcal{U}' = \{U_j \mid j \in J\}, \quad J \subseteq I$$

such that $\bigcup_{j \in J} U_j = X$.

If J has finitely many elements, we say $\mathcal{U}'$ is a finite subcover of X.

We say X is compact if any open cover of X has a finite subcover. ■

R If $A \subseteq X$ has a subspace topology. then A is compact iff for any open collection of open sets (in X) $\{U_i\}$ such that $A \subseteq \bigcup_{i \in I} U_i$, there exists a fintie subcover $A \subseteq \bigcup_{k=1}^{n} U_{i_k}$.

Proposition 4.15 Let X be a topological space. The followings are equivalent:

1. The space X is compact
2. If $\{V_i \mid i \in I\}$ is a collection of closed subsets in X such that

$$\bigcap_{j \in J} V_j \neq \emptyset, \quad \text{for all finite } J \subseteq I,$$

   then $\cap_{i \in I} V_i \neq \emptyset$.

Compactness is an **intrisical** property, i.e., we do not need to worry about which underlying space for this definition.

■ **Example 4.6** 1. $X \subseteq \mathbb{R}^n$ is compact iff X is closed and bounded. (Heine-Borel)

2. Let $K \subseteq \mathbb{R}^n$ be compact, then define the set

$$\mathcal{C}(K) = \{\text{all continuous mapping } f : K \to \mathbb{R}\}$$

   Note that the d_∞ metric associated with $\mathcal{C}(K)$, say $\|f\|_\infty = \sup_{k \in K} f(k)$, is well-defined.

   Under the metric space $(\mathcal{C}(K), d_\infty)$, any $\mathcal{J} \subseteq \mathcal{C}(K)$ is compact, if and only if $\mathcal{J}$ is closed, bounded, and equi-continuous. (Aresul-Ascoli)

Therefore, we can see that the compactness is not equivalent to the closedness together with boundedness. ■

Proposition 4.16 Let X be a compact space, then all closed subset $A \subseteq X$ are compact.

Proof. Let $\{V_i \mid i \in I\}$ be a collection of closed subsets in A such that

$$\cap_{j \in J} V_j \neq \emptyset, \quad \text{for any finite } J \subseteq I.$$

As A is closed in X, we imply V_j is closed in X.

Due to the compactness of X and proposition (4.15), we imply

$$\cap_{i \in I} V_i \neq \emptyset$$

By the reverse direction of proposition (4.15), we imply A is compact. ■

R Now consider the reverse direction of proposition (4.16), i.e., are all compact subsets $K \subseteq X$ closed in X?

In general, the converse does not hold. Note that $K = \{x\}$ is compact for any topology X. However, there are some topologies such that $\{x\}$ is closed.

In order to obtain the converse of proposition (4.16), we need to obtain another **separation axiom**:

Proposition 4.17 Let X be Hausdorff, $K \subseteq X$ be compact, and $x \in X \setminus K$. Then there exists open $U, V \subseteq X$ such that $U \cap V = \emptyset$ and

$$U \cap V = \emptyset, \quad K \subseteq U, \quad x \in V.$$

Proof. Let $k \in K$, then by Hausdorffness, there exists open $U_k \ni k, V_k \ni x$ such that $U_k \cap V_k = \emptyset$. Therefore, $\{U_k\}_{k \in K}$ forms an open cover of K. By compactness of K, $\{U_{k_i}\}_{i=1}^n$ covers K. Constructing the set

$$U := \bigcup_{i=1}^n U_{k_i}, \quad V = \bigcap_{i=1}^n V_{k_i}$$

gives the desired result. ■

By making use of this separation axiom, we obtain the converse of proposition (4.16):

Corollary 4.3 All compact K in Hausdorff X is closed.

Proof. For $\forall x \in X \setminus K$, by proposition (4.17) there exists open V such that $x \in V \subseteq X \setminus K$, and therefore $X \setminus K$ is open. ■