4.3. Monday for MAT4002

There will be a quiz next Monday. The scope is everything before CNY holiday. There will be one question with four parts for 40 minutes.

4.3.1. Hausdorffness

Reviewing. A topological space $(X, \mathcal{T})$ is said to be **Hausdorff** (or satisfy the second separtion property), if given any distinct points $x, y \in X$, there exist disjoint open sets U, V such that $U \ni x$ and $V \ni y$.

Proposition 4.5 If the topological space $(X, \mathcal{T})$ is Hausdorff, then all sequences $\{x_n\}$ in X has at most one limit.

Proof. Suppose on the contrary that

$$\{x_n\} \to a, \quad \{x_n\} \to b, \text{ with } a \neq b$$

By separation property, there exists $U, V \in \mathcal{T}$ and $U \cap V = \emptyset$ such that $U \ni a$ and $V \ni b$.

By tje openness of U, there exists N such that $\{x_N, x_{N+1}, \dots\} \subseteq U$, since $\{x_n\} \to a \in U$. Similarly, there exists M such that $\{x_M, x_{M+1}, \dots\} \subseteq V$. Take $K = \max\{M, N\} + 1$, then $\emptyset \neq U \cap V \ni x_K$, which is a contradiction. ∎

Proposition 4.6 Let X, Y be Hausdorff spaces. Then $X \times Y$ is Hausdorff with product topology.

Proof. Suppose that $(x_1, y_1) \neq (x_2, y_2)$ in $X \times Y$. Then $x_1 \neq x_2$ or $y_1 \neq y_2$. w.l.o.g., assume that $x_1 \neq x_2$, then there exists U, V open in X such that $x_1 \in U, x_2 \in V$ with $U \cap V = \emptyset$.

Therefore, we imply $(U \times Y), (V \times Y) \in \mathcal{T}_{X \times Y}$, and

$$(U \times Y) \cap (V \times Y) = (U \cap V) \cap Y = \emptyset$$

with $(x_1, y_1) \in U \times Y, (x_2, y_2) \in V \times Y$, i.e., $X \times Y$ is Hausdorff with product topology. ∎

The same argument applies if the second separation property is replaced by first separation property.

Proposition 4.7 If $f : X \to Y$ is an injective continuous mapping, then Y is Hausdorff implies X is Hausdorff.

Proof. Suppose that Y satisfies the second separation property. For given $a \neq b$ in X, we imply $f(a) \neq f(b)$ in Y. Therefore, there exists $U \ni f(a), V \ni f(b)$ with $U \cap V = \emptyset$. It follows that

$$a \in f^{-1}(U), b \in f^{-1}(V), \quad f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset,$$

i.e., X is Hausdorff. ∎

Corollary 4.1 If $f : X \to Y$ is homeomorphic, then X is Hausdorff iff Y is Hausdorff, i.e., Hausdorffness is a topological property (i.e., a property that is preserved under homeomorphism).

4.3.2. Connectedness

Definition 4.4 [Connected] The topological space $(X, \mathcal{T})$ is **disconnected** if there are open $U, V \in \mathcal{T}$ such that

$$U \neq \emptyset, V \neq \emptyset, \quad U \cap V = \emptyset, \quad U \cup V = X. \tag{4.4}$$

If no such $U, V \in \mathcal{T}$ exist, then X is **connected**. ■

Proposition 4.8 Let $(X, \mathcal{T})$ be topological spaces. TFAE (i.e., the followings are equivalent):

1. X is connected
2. The **only** subset of X which are both open and closed are $\emptyset$ and X
3. Any continuous function $f : X \to \{0, 1\}$ ($\{0, 1\}$ is equipped with discrete topology) is a constant function.

Proof. (1) implies (2): Suppose that $U \subseteq X$ is both open and closed. Then $U, X \setminus U$ are both open and disjoint, and $U \cup (X \setminus U) = X$. By connnectedness, either $U = \emptyset$ or $X \setminus U = \emptyset$. Therefore, $U = \emptyset$ or X.

(2) implies (3): Note that $U = f^{-1}(\{0\})$ and $V = f^{-1}(\{1\})$ are open disjoint sets in X satisfying $U \cup V = X$. By the connectedness of X, either $(U,V) = (X,\emptyset)$ or $(V,U) = (\emptyset, X)$. In either case, we imply f is a constant function.

(3) implies (2): Suppose that $U \subseteq X$ is both open and closed. Construct the mapping

$$f(x) = \begin{cases} 0, & x \in U \\ 1, & x \in X \setminus U \end{cases}$$

It's clear that f is continuous, and therefore $f(x) = 0$ or 1. Therefore $U = \emptyset$ or X.

(2) implies (1): Suppose on the contrary that there exists open U, V such that (4.4) holds. By (4.4), we imply $U = X \setminus V$ is closed as well. Since $U \neq \emptyset$ and $U = \emptyset$ or X, we imply $U = X$, which implies $V = \emptyset$, which is a contradiction. ■

Corollary 4.2 The interval $[a,b] \subseteq \mathbb{R}$ is connnected

Proof. Suppose on the contrary that there exists continuous function $f : [a,b] \to \{0,1\}$ that takes 2 values. Construct the mapping $\tilde{f} : [a,b] \to \mathbb{R}$

$$\begin{aligned} &\tilde{f} : [a,b] \xrightarrow{f} \{0,1\} \xrightarrow{i} \mathbb{R}, \\ \text{with} \quad &\tilde{f} = i \circ f. \end{aligned}$$

Note that $\{0,1\} \subseteq \mathbb{R}$ denotes the subspace topology, we imply the inclusion mapping $i : \{0,1\} \to \mathbb{R}$ with $s \mapsto s$ is continuous. The composition of continuous mappings is continuous as well, i.e., $\tilde{f}$ is continuous.

Since the function f can take two values, there exists $p, q \in [a,b]$ such that $\tilde{f}(p) = i \circ f(p) = 0$ and $\tilde{f}(q) = i \circ f(q) = 1$. By intermediate value theorem, there exists $r \in [a,b]$ such that $\tilde{f}(r) = i \circ f(r) = 1/2$, which implies $f(r) = \frac{1}{2}$, which is a contradiction. ■

Definition 4.5 [Connected subset] A non-empty subset $S \subseteq X$ is **connected** if S with the subspace topology is connected

Equivalently, $S \subseteq X$ is connected if, whenever U, V are open in X such that $S \subseteq U \cup V$, and $(U \cap V) \cap S = \emptyset$, one can imply either $U \cap S = \emptyset$ or $V \cap S = \emptyset$.

Proposition 4.9 If $f : X \to Y$ is continuous mapping, and the subset $A \subseteq X$ is connected, then $f(A)$ is connected. In other words, the continuous image of a connected set is connected.

Proof. Suppose that $U, V \subseteq Y$ is open such that

$$f(A) \subseteq U \cup V, \quad (U \cap V) \cap f(A) = \emptyset.$$

Therefore we imply

$$A \subseteq f^{-1}(U) \cup f^{-1}(V), \quad (f^{-1}(U) \cap A) \cap (f^{-1}(V) \cap A) = \emptyset$$

By connectedness of A, either $f^{-1}(U) \cap A = \emptyset$ or $f^{-1}(V) \cap A = \emptyset$. Therefore, $f(A) \cap U = \emptyset$ or $f(A) \cap V = \emptyset$, i.e., $f(A)$ is connected. ∎

Proposition 4.10 If $\{A_i\}_{i \in I}$ are connected and $A_i \cap A_j \neq \emptyset$ for $\forall i, j \in I$, then the set $\bigcup_{i \in I} A_i$ is connected.

Proof. Suppose the function $f : \cup_{i \in I} A_i \to \{0, 1\}$ is a continuous map. Then we imply that its restriction $f|_{A_i} = f \circ i : A_i \to \{0, 1\}$ is continuous for all $i \in I$. Thus $f|_{A_i}$ is a constant for all $i \in I$. Due to the non-empty intersection of A_i, A_j for $\forall i, j \in I$, we imply f is constant. ∎

Proposition 4.11 If X, Y are connnected, then $X \times Y$ is connected using product topology.

Proof. It's clear that $X \times \{y_0\}$ is connected in $X \times Y$ for fixed y_0; and $\{x_0\} \times Y$ is connected for fixed x_0.

Therefore, for fixed $y_0 \in Y$, construct $B = X \times \{y_0\}$ and $C_x = \{x\} \times Y$, which follows that

$$B \cap C_x = \{(x, y_0)\} \neq \emptyset, \forall x \in X \implies B \cup \left\{ \bigcup_{x \in X} C_x \right\} = X \times Y \text{ is connected.}$$

■

Definition 4.6 [Path Connectes] Let $(X, \mathcal{T})$ be a topological space.

1. A path connecting 2 points $x, y \in X$ is a continuous function $\tau : [0,1] \to X$ with $\tau(0) = x, \tau(1) = y$.
2. X is path-connected if any 2 points in X can be connected by a path.
3. The set $A \subseteq X$ is path-connected, if A sastisfies the condition using subspace topology.

   Or equivalently, A is path-connected if for any 2 points in X, there exists a continuous $t : [0,1] \to X$ with $t(x) \in A$ for any x, connecting the 2 points.