3.6. Wednesday for MAT4002

3.6.1. Remarks on product space

Reviewing.

- Product Topology: For topological space $(X, \mathcal{T}_X)$ and $(Y, \mathcal{Y})$, define the basis

$$\mathcal{B}_{X\times Y} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$$

  and the family of union of subsets in $\mathcal{B}_{X\times Y}$ forms a product topology.

Proposition 3.9 a ring torus is homeomorphic to the Cartesian product of two circles, say $S^1 \times S^1 \cong T$.

Proof. Define a mapping $f : [0, 2\pi] \times [0, 2\pi] \to T$ as

$$f(\theta, \phi) = \Big((R + r\cos\theta)\cos\phi, \quad (R + r\cos\theta)\sin\phi, \quad r\sin\theta\Big)$$

Define $i : T \to \mathbb{R}^3$, we imply

$$i \circ f : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3 \text{ is continuous}$$

Therefore we imply $f : [0, 2\pi] \times [0, 2\pi] \to T$ is continuous. Together with the condition that

$$\begin{cases} f(0, y) = f(2\pi, y) \\ f(x, 0) = f(x, 2\pi) \end{cases}$$

we imply the function $f : S^1 \to S^1 \to T$ is continuous. We can also show it is bijective. We can also show f^{-1} is continuous. ■

Proposition 3.10 1. Let $X \times Y$ be endowed with product topology. The projection mappings defined as

$$p_X : X \times Y \to X, \text{ with } p_X(x,y) = x$$
$$p_Y : X \times Y \to Y, \text{ with } p_Y(x,y) = y$$

are continuous.

2. (an equivalent definition for product topology) The product topology is the **coarest topology** on $X \times Y$ such that p_X and p_Y are both continuous.
3. (an equivalent definition for product topology) Let Z be a topological space, then the product topology is the unique topology that the red and the blue line in the diagram commutes:

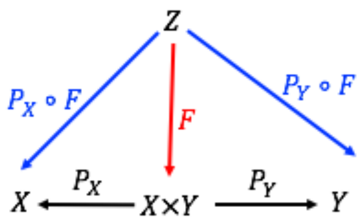

Figure 3.3: Diagram summarizing the statement (*)

namely,

the mapping $F : Z \to X \times Y$ is continuous iff both $P_X \circ F : Z \to X$ and $P_Y \circ F : Z \to Y$ are continuous. (*)

Proof. 1. For any open U, we imply $p_X^{-1}(U) = U \times Y \in \mathcal{B}_{X\times Y} \subseteq \mathcal{T}_{X\times Y}$, i.e., $p_X^{-1}(U)$ is open. The same goes for p_Y.

2. It suffices to show any topology $\mathcal{T}$ that meets the condition in (2) must contain $\mathcal{T}_{\text{product}}$. We imply that for $\forall U \in \mathcal{T}_X, V \in \mathcal{T}_Y$,

$$\begin{cases} p_X^{-1}(U) = U \times X \in \mathcal{T} \\ p_Y^{-1}(V) = X \times V \in \mathcal{T} \end{cases} \implies (U \times Y) \cap (X \times V) = (U \cap X) \times (Y \cap V) = U \times V \in \mathcal{T},$$

which implies $\mathcal{B}_{X\times Y} \subseteq \mathcal{T}$. Since $\mathcal{T}$ is closed for union operation on subsets, we imply $\mathcal{T}_{\text{product topology}} \subseteq \mathcal{T}$.

3. (a) Firstly show that $\mathcal{T}_{\text{product}}$ satisfies (*).

    - For the forward direction, by (1) we imply both $p_X \circ F$ and $p_Y \circ F$ are continuous, since the composition of continuous functions are continuous as well.
    - For the reverse direction, for $\forall U \times \mathcal{T}_X, V \in \mathcal{T}_Y$,

    $$F^{-1}(U \times V) = (p_X \circ F)^{-1}(X) \cap (p_Y \circ F)^{-1}(Y),$$

    which is open due to the continuity of $p_X \circ F$ and $p_Y \circ F$.

   (b) Then we show the uniqueness of $\mathcal{T}_{\text{product}}$. Let $\mathcal{T}$ be another topology $X \times Y$ satisfying (*).

    - Take $Z = (X \times Y, \mathcal{T})$, and consider the identity mapping $F = \text{id} : Z \to Z$, which is continuous. Therefore $p_X \circ \text{id}$ and $p_Y \circ \text{id}$ are continuous, i.e., p_X and p_Y are continuous. By (2) we imply $\mathcal{T}_{\text{product}} \subseteq \mathcal{T}$.
    - Take $Z = (X \times Y, \mathcal{T}_{\text{product}})$, and consider the identity mapping $F = \text{id} : Z \to Z$. Note that $p_X \circ F = p_X$ and $p_Y \circ F = p_Y$, which is continuous by (1). Therefore, the identity mapping $F : (X \times Y, \mathcal{T}_{\text{product}}) \to (X \times Y, \mathcal{T})$ is continuous, which implies

    $$U = \text{id}^{-1}(U) \subseteq \mathcal{T}_{\text{product}} \text{ for } \forall U \in \mathcal{T},$$

    i.e., $\mathcal{T} \subseteq \mathcal{T}_{\text{product}}$.

   The proof is complete.

■

Definition 3.6 [Disjoint Union] Let $X \times Y$ be two topological spaces, then the **disjoint union** of X and Y is

$$X \sqcup Y := (X \times \{0\}) \cup (Y \times \{1\})$$

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1. We define that U is open in $X \sqcup Y$ if

   (a) $U \cap (X \times \{0\})$ is open in $X \times \{0\}$; and

   (b) $U \cap (Y \times \{1\})$ is open in $Y \times \{1\}$.

   We also need to show the weill-definedness for this definition.

2. S is open in $X \perp Y$ iff S can be expressed as

$$S = (U \times \{0\}) \cup (V \times \{1\})$$

   where $U \subseteq X$ is open and $V \subseteq Y$ is open.

3.6.2. Properties of Topological Spaces

3.6.2.1. Hausdorff Property

Definition 3.7 [First Separation Axiom] A topological space X satisfies the **first separation axiom** if for any two distinct points $x \neq y \in X$, there exists open $U \ni x$ but not including y.

Proposition 3.11 A topological space X has first separation property if and only if for $\forall x \in X$, $\{x\}$ is closed in X.

Proof. Sufficiency. Suppose that $x \neq y$, then construct $U := X \setminus \{y\}$, which is a open set that contains x but not includes y.

Necessity. Take any $x \in X$, then for $\forall y \neq x$, there exists $y \in U_y$ that is open and $x \notin U_y$. Thus

$$\{y\} \subseteq U_y \subseteq X \setminus \{x\}$$

which implies

$$\bigcup_{y \in X \setminus \{x\}} \{y\} \subseteq \bigcup_{y \in X \setminus \{x\}} U_y \subseteq X \setminus \{x\},$$

i.e., $X \setminus \{x\} = \bigcup_{y \in X \setminus \{x\}} U_y$ is open in X, i.e., $\{x\}$ is closed in X. ■

Definition 3.8 [Second separation Axiom] A topological space satisfies the **second separation axiom** (or X is Hausdorff) if for all $x \neq y$ in X, there exists open sets U, V such that

$$x \in U, \quad y \in V, \quad U \cap V = \varnothing$$

■

■ **Example 3.13** All metrizable topological spaces are Hausdorff.

Suppose $d(x,y) = r > 0$, then take $B_{r/2}(x)$ and $B_{r/2}(y)$ ■

■ **Example 3.14** Note that a topological space that is **first separable** may not necessarily be **second separable**:

Consider $\mathcal{T}_{\text{co-finite}}$, then X is first separable but not Hausdorff:

Suppose on the contrary that for given $x \neq y$, there exists open sets U, V such that $x \in U, y \in V$, and

$$U \cap V = \varnothing \implies X = X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V),$$

implying that the union of two finite sets equals X, which is infinite, which is a contradiction.

■