2.6. Wednesday for MAT4002

Reviewing.

1. Interior, Closure:

$$\overline{A} = \{x \mid \forall U \ni x \text{ open}, U \bigcap A \neq \emptyset\}$$

2. Accumulation points

2.6.1. Remark on Closure

Definition 2.14 [Sequential Closure] Let A_S be the set of limits of any convergent sequence in A, then A_S is called the **sequential closure** of A.

Definition 2.15 [Accumulation/Cluster Points] The set of accumulation (limit) points is defined as

$$A' = \{x \mid \forall U \ni x \text{ open }, (U \setminus \{x\}) \bigcap A \neq \emptyset\}$$

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1. (a) There exists some point in A but not in A':

$$A = \{1, 2, 3, \dots, n, \dots\}$$

   Then any point in A is not in A'

   (b) There also exists some point in A' but not in A:

$$A = \{\frac{1}{n} \mid n \geq 1\}$$

   Then the point 0 is in A' but not in A.

2. The closure $\overline{A} = A \bigcup A'$.
3. The size of the sequentical closure A_S is between A and $\overline{A}$, i.e., $A \subseteq A_S \subseteq \overline{A}$:

It's clear that $A \subseteq A_S$, since the sequence $\{a_n := a\}$ is convergent to a for $\forall a \in A$.

For all $a \in A_S$, we have $\{a_n\} \to a$. Then for any open $U \ni a$, there exists N such that $\{a_N, a_{N+1}, \dots\} \subseteq U \bigcap A \neq \emptyset$. Therefore, $a \in \overline{A}$, i.e., $A_S \subseteq \overline{A}$.

Question: Is $A_S = \overline{A}$?

Proposition 2.20 Let (X, d) be a metric space, then $A_S = \bar{A}$.

Proof. Let $a \in \overline{A}$, then there exists $a_n \in B_{1/n}(a) \bigcap A$, which implies $\{a_n\} \to a$, i.e., $a \in A_S$. ∎

R If $(X, \mathcal{T})$ is metrizable, then $A_S = \overline{A}$. The same goes for first countable topological spaces. However, A_S is a proper subset of A in general:

Let $A \subseteq X$ be the set of continuous functions, where $X = \mathbb{R}^{\mathbb{R}}$ denotes the set of all real-valued functions on $\mathbb{R}$, with the topology of pointwise convergence.

Then $A_S = B_1$, the set of all functions of first Baire-Category on $\mathbb{R}$; and $[A_S]_S = B_2$, the set of all functions of second Baire-Category on $\mathbb{R}$. Since $B_1 \neq B_2$, we have $[A_S]_S = A_S$. Note that $\overline{\overline{A}} = \overline{A}$. We conclude that A_S cannot equal to $\overline{A}$, since the sequential closure operator cannot be idemotenet.

Definition 2.16 [Boundary] The **boundary of A** is defined as

$$\partial \boldsymbol{A} = \overline{A} \setminus A^{\circ}$$

Proposition 2.21 Let $(X, \mathcal{T})$ be a topological space with $A, B \subseteq X$.

$$\overline{X \setminus A} = X \setminus A^{\circ}, \quad (X \setminus B)^{\circ} = X \setminus \overline{B} \quad \partial A = \overline{A} \bigcap (\overline{X \setminus A})$$

Proof.

$$X \setminus A^\circ = X \setminus \left(\bigcup_{U \text{ is open}, U \subseteq A} U \right) \tag{2.2a}$$

$$= \bigcap_{U \text{ is open}, U \subseteq A} (X \setminus U) \tag{2.2b}$$

$$= \bigcap_{V \text{ is closed}, F \supseteq X \setminus A} F \tag{2.2c}$$

$$= \overline{X \setminus A} \tag{2.2d}$$

Denoting $X \setminus A$ by B, we obtain:

$$(X \setminus B)^\circ = A^\circ \tag{2.3a}$$

$$= X \setminus (X \setminus A^\circ) \tag{2.3b}$$

$$= X \setminus \overline{X \setminus A} \tag{2.3c}$$

$$= X \setminus \overline{B} \tag{2.3d}$$

By definition of ∂A,

$$\partial A = \overline{A} \setminus A^\circ \tag{2.4a}$$

$$= \overline{A} \bigcap (X \setminus A^\circ) \tag{2.4b}$$

$$= \overline{A} \bigcap (\overline{X \setminus A}) \tag{2.4c}$$

■

2.6.2. Functions on Topological Space

Definition 2.17 [Continuous] Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a map. Then the function f is continuous, if

$$U \in \mathcal{T}_Y \implies f^{-1}(U) \in \mathcal{T}_X$$

■ **Example 2.16**

1. The identity map $\mathrm{id} : (X, \mathcal{T}) \to (X, \mathcal{T})$ defined as $x \mapsto x$ is continuous
2. The identity map $\mathrm{id} : (X, \mathcal{T}_{\text{discrete}}) \to (X, \mathcal{T}_{\text{indiscrete}})$ defined as $x \mapsto x$ is continuous. Since $\mathrm{id}^{-1}(\varnothing) = \varnothing$ and $\mathrm{id}^{-1}(X) = X$
3. The identity map $\mathrm{id} : (X, \mathcal{T}_{\text{indiscrete}}) \to (X, \mathcal{T}_{\text{discrete}})$ defined as $x \mapsto x$ is not continuous.

Proposition 2.22 If $f : X \to Y$, and $g : Y \to Z$ be continuous, then $g \circ f$ is continuous

Proof. For given $U \in \mathcal{T}_Z$, we imply

$$g^{-1}(U) \in \mathcal{T}_Y \implies f^{-1}(g^{-1}(U)) \in \mathcal{T}_X,$$

i.e., $(g \circ f)^{-1}(U) \in \mathcal{T}_X$ ■

Proposition 2.23 Suppose $f : X \to Y$ is continuous between two topological spaces. Then $\{x_n\} \to X$ implies $\{f(x_n)\} \to f(x)$.

Proof. Take open $U \ni f(x)$, which implies $f^{-1}(U) \ni x$. Since $f^{-1}(U)$ is open, we imply there exists N such that

$$\{x_n \mid n \geq N\} \subseteq f^{-1}(U),$$

i.e., $\{f(x_n) \mid n \geq N\} \subseteq U$ ■

We use the notion of Homeomorphism to describe the equivalence between two topological spaces.

Definition 2.18 [Homeomorphism] A **homeomorphism** between spaces topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is a bijection

$$f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y),$$

such that both f and f^{-1} are continuous.

2.6.3. Subspace Topology

Definition 2.19 Let $A \subseteq X$ be a non-empty set. The **subspace topology** of A is defined as:

1. $\mathcal{T}_A := \{U \bigcap A \mid U \in \mathcal{T}_A\}$
2. The **coarsest** topology on A such that the **inclusion map**

$$i : (A, \mathcal{T}_A) \to (X, \mathcal{T}_X), \quad i(x) = x$$

   is continuous.

   (We say the topology $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$, or $\mathcal{T}_2$ is **finer** than $\mathcal{T}_1$, if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ e.g., $\mathcal{T}_{\text{discrete}}$ is the finest topology, and $\mathcal{T}_{\text{indiscrete}}$ is coarsest topology.)
3. The (**unique**) topology such that for any $(Y, \mathcal{T}_Y)$,

$$f : (Y, \mathcal{T}_Y) \to (A, \mathcal{T}_A)$$

   is continuous iff $i \circ f : (Y, \mathcal{T}_Y) \to (X, \mathcal{T}_X)$ (where i is the inclusion map) is continuous.

Proposition 2.24 The definition (1) and (2) in (2.19) are equivalent.

Outline. The proof is by applying

$$i^{-1}(S) = S \bigcap A, \quad \forall S$$

■

Example 2.17 Let all English and numerical letters be subset of $\mathbb{R}^2$:

P,6

The homeomorphism can be constrcuted between these two English letters.

Proposition 2.25 The definition (2) and (3) in (2.19) are equivalent.

Proof. Necessity.

- For $\forall U \in \mathcal{T}_X$, consider that

$$(i \circ f)^{-1}(U) = f^{-1}(i^{-1}(U)) = f^{-1}(U \bigcap A)$$

  since $U \bigcap A \in \mathcal{T}_A$ and f is continuous, we imply $(i \circ f)^{-1}(U) \in \mathcal{T}_Y$

- For $\forall U' \in \mathcal{T}_A$, we have $U' = U \bigcap A$ for some $U \in \mathcal{T}_X$. Therefore,

$$f^{-1}(U') = f^{-1}(U \bigcap A) = f^{-1}(i^{-1}(U)) = (i \circ f)^{-1}(U) \in \mathcal{T}_Y.$$

The sufficiency is left as exercise. ■

Proposition 2.26
1. The definition (1) in (2.19) does define a topology of A
2. Closed sets of A under subspace topology are of the form $V \bigcap A$, where V is closed in X

Proposition 2.27 Suppose $(A, \mathcal{T}_A) \subseteq (X, \mathcal{T}_X)$ is a subspace topology, and $B \subseteq A \subseteq X$. Then

1. $\bar{B}^A = \bar{B}^X \bigcap A$.
2. $B^{\circ A} \supseteq B^{\circ X}$

Proof. By proposition (2.26), $\bar{B}^X \bigcap A$ is closed in A, and $\bar{B}^X \bigcap A \supset B$, which implies

$$\bar{B}^A \subseteq \bar{B}^X \bigcap A$$

Note that $\bar{B}^A \supset B$ is closed in A, which implies $\bar{B}^A = V \bigcap A \subseteq V$, where V is closed in X. Therefore,

$$\bar{B}^X \subseteq V \implies \bar{B}^X \bigcap A \subseteq V \bigcap A = \bar{B}^A$$

Therefore, $\bar{B}^A = \bar{B}^X \subseteq V$

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Can we have $B^{\circ X} = B^{\circ A}$?

2.6.4. Basis (Base) of a topology

Roughly speaking, a basis of a topology is a family of "generators" of the topology.

Definition 2.20 Let $(X, \mathcal{T})$ be a topological space. A family of subsets $\mathcal{B}$ in X is a **basis** for $\mathcal{T}$ if

1. $\mathcal{B} \subseteq \mathcal{T}$, i.e., everything in $\mathcal{B}$ is open
2. Every $U \in \mathcal{T}$ can be written as union of elements in $\mathcal{B}$.

■ **Example 2.18** 1. $\mathcal{B} = \mathcal{T}$ is a basis.

2. For $X = \mathbb{R}^n$,

$$\mathcal{B} = \{B_r(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{Q}^n, r \in \mathbb{Q} \bigcap (0, \infty)\}$$

Exercise: every $(a, b) = \bigcup_{i \in I}(p_i, q_i)$ for $p_i, q_i \in \mathbb{Q}$.

Therefore, $\mathcal{B}$ is countable.

Proposition 2.28 If $(X, \mathcal{T})$ has a countable basis, e.g., $\mathbb{R}^n$, then $(X, \mathcal{T})$ has a second-countable space.