1.6. Wednesday for MAT4002

Reviewing.

- Metric Space (X,d)
- Open balls and open sets (note that the emoty set $\emptyset$ is open)
- Define the collection of open sets in X, say $\mathcal{T}$ is the topology.

Exercise.

1. Show that the $\mathcal{T}_2$ under $(X=\mathbb{R}^2,d_2)$ and $\mathcal{T}_\infty$ under $(X=\mathbb{R}^2,d_\infty)$ are the same.

   Ideas. Follow the procedure below:

   An open ball in d_2-metric is open in d_∞;

   Any open set in d_2-metric is open in d_∞;

   Switch d_2 and d_∞. ■

2. Describe the topology $\mathcal{T}_{\text{discrete}}$ under the metric space $(X=\mathbb{R}^2,d_{\text{discrete}})$.

   Outlines. Note that $\{x\}=B_{1/2}(x)$ is an open set.

   For any subset $W\subseteq\mathbb{R}^2$, $W=\bigcup_{w\in W}\{w\}$ is open.

   Therefore $\mathcal{T}_{\text{discrete}}$ is all subsets of $\mathbb{R}^2$. ■

1.6.1. Forget about metric

Next, we will try to define closedness, compactness, etc., without using the tool of metric:

Definition 1.18 [closed] A subset $V\subseteq X$ is **closed** if $X\setminus V$ is **open**.

Example 1.19 Under the metric space $(\mathbb{R},d_1)$,

$$\mathbb{R}\setminus[b,a]=(a,\infty)\bigcup(-\infty,b)\text{ is open}\implies[b,a]\text{ is closed}$$

Proposition 1.14 Let X be a metric space.

1. $\emptyset, X$ is closed in X
2. If F_α is closed in X, so is $\bigcap_{\alpha\in A} F_\alpha$.
3. If $F_1,\dots,F_k$ is closed, so is $\bigcup_{i=1}^k F_i$.

Proof. 1. Note that X is open in X, which implies $\emptyset = X \setminus X$ is closed in X; Similarly, $\emptyset$ is open in X, which implies $X = X\setminus\emptyset$ is closed in X;

2. The set F_α is closed implies there exists open $U_\alpha \subseteq X$ such that $F_\alpha = X\setminus U_\alpha$. By De Morgan's Law,

$$\bigcap_{\alpha\in A} F_\alpha = \bigcap_{\alpha\in A}(X\setminus U_\alpha) = X\setminus(\bigcup_{\alpha\in A} U_\alpha).$$

By part (a) in proposition (1.6), the set $\bigcup_{\alpha\in A} U_\alpha$ is openm which implies $\bigcap_{\alpha\in A} F_\alpha$ is closed.

3. The result follows from part (b) in proposition (1.6) by taking complements.

■

We illustrate examples where open set is used to define convergence and continuity.

1. Convergence of sequences:

Definition 1.19 [Convergence] Let (X,d) be a metric space, then $\{x_n\}\to x$ means

$$\forall\varepsilon>0, \exists N \text{ such that } d(x_n,x)<\varepsilon, \forall n\ge N.$$

We will study the convergence by using open sets instead of metric.

Proposition 1.15 Let X be a metric space, then $\{x_n\}\to x$ if and only if for $\forall$ open set $U\ni x$, there exists N such that $x_n\in U$ for $\forall n\ge N$.

Proof. Necessity: Since $U\ni x$ is open, there exists $\varepsilon>0$ such that $B_\varepsilon(x)\subseteq U$. Since $\{x_n\}\to x$, there exists N such that $d(x_n,x)<\varepsilon$, i.e., $x_n\in B_\varepsilon(x)\subseteq U$ for $\forall n\ge N$.

Sufficiency: Let $\varepsilon > 0$ be given. Take the open set $U = B_\varepsilon(x) \ni x$, then there exists N such that $x_n \in U = B_\varepsilon(x)$ for $\forall n \geq N$, i.e., $d(x_n, x) < \varepsilon$, $\forall n \geq N$.

■

2. Continuity:

Definition 1.20 [Continuity] Let (X, d) and (Y, ρ) be given metric spaces. Then $f : X \to Y$ is continuous at $x_0 \in X$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d(x, x_0) < \delta \implies \rho(f(x), f(x_0)) < \varepsilon.$$

The function f is continuous on X if f is continous for all $x_0 \in X$.

We can get rid of metrics to study continuity:

Proposition 1.16 (a) The function f is continuous at x if and only if for all open $U \ni f(x)$, there exists $\delta > 0$ such that the set $B(x, \delta) \subseteq f^{-1}(U)$.

(b) The function f is continuous on X if and only if $f^{-1}(U)$ is open in X for each open set $U \subseteq Y$.

During the proof we will apply a small lemma:

Proposition 1.17 f is continuous at x if and only if for all $\{x_n\} \to x$, we have $\{f(x_n)\} \to f(x)$.

Proof. (a) *Necessity*:

Due to the openness of $U \ni f(x)$, there exists a ball $B(f(x), \varepsilon) \subseteq U$.

Due to the continuity of f at x, there exists $\delta > 0$ such that $d(x, x') < \delta$ implies $d(f(x), f(x')) < \varepsilon$, which implies

$$f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \subseteq U,$$

which implies $B(x, \delta) \subseteq f^{-1}(U)$.

Sufficiency:

Let $\{x_n\} \to x$. It suffices to show $\{f(x_n)\} \to f(x)$. For each open $U \ni f(x)$, by hypothesis, there exists $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(U)$.

Since $\{x_n\} \to x$, there exists N such that

$$x_n \in B_\delta(x) \subseteq f^{-1}(U), \forall n \geq N \implies f(x_n) \in U, \forall n \geq N$$

Let $\varepsilon > 0$ be given, and then construct the $U = B_\varepsilon(f(x))$. The argument above shows that $f(x_n) \in B_\varepsilon(f(x))$ for $\forall n \geq N$, which implies $\rho(f(x_n), f(x)) < \varepsilon$, i.e., $\{f(x_n)\} \to f(x)$.

(b) For the forward direction, it suffices to show that each point x of $f^{-1}(U)$ is an interior point of $f^{-1}(U)$, which is shown by part (a); the converse follows trivially by applying (a).

■

R As illustracted above, convergence, continuity, (and compactness) can be defined by using open sets $\mathcal{T}$ only.

1.6.2. Topological Spaces

Definition 1.21 A **topological space** $(X, \mathcal{T})$ consists of a (non-empty) set X, and a family of subsets of X ("open sets" $\mathcal{T}$) such that

1. $\emptyset, X \in \mathcal{T}$
2. $U, V \in \mathcal{T}$ implies $U \bigcap V \in \mathcal{T}$
3. If $U_\alpha \in \mathcal{T}$ for all $\alpha \in \mathcal{A}$, then $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{T}$.

The elements in $\mathcal{T}$ are called **open subsets** of X. The $\mathcal{T}$ is called a **topology** on X. ■

■ **Example 1.20** 1. Let (X, d) be any metric space, and

$$\mathcal{T} = \{\text{all open subsets of } X\}$$

It's clear that $\mathcal{T}$ is a topology on X.

2. Define the discrete topology

$$\mathcal{T}_{\text{dis}} = \{\text{all subsets of } X\}$$

It's clear that $\mathcal{T}_{\text{dis}}$ is a topology on X, (which also comes from the discrete metric (X, d_{discrete})).

R We say $(X, \mathcal{T})$ is induced from a metric (X, d) (or it is **metrizable**) if $\mathcal{T}$ is the faimly of open subsets in (X, d).

3. Consider the indiscrete topology $(X, \mathcal{T}_{\text{indis}})$, where X contains more than one element:

$$\mathcal{T}_{\text{indis}} = \{\emptyset, X\}.$$

Question: is $(X, \mathcal{T}_{\text{indis}})$ metrizable? No. For any metric d defined on X, let x, y be distinct points in X, and then $\varepsilon := d(x, y) > 0$, hence $B_{\frac{1}{2}\varepsilon}(x)$ is a open set belonging to the corresponding induced topology. Since $x \in B_{\frac{1}{2}\varepsilon}(x)$ and $y \notin B_{\frac{1}{2}\varepsilon}(y)$, we conclude that $B_{\frac{1}{2}\varepsilon}(x)$ is neither $\emptyset$ nor X, i.e., the topology induced by any metric d is not the indiscrete topology.

4. Consider the cofinite topology $(X, \mathcal{T}_{\text{cofin}})$:

$$\mathcal{T}_{\text{cofin}} = \{U \mid X \setminus U \text{ is a finite set}\} \bigcup \{\emptyset\}$$

Question: is $(X, \mathcal{T}_{\text{cofin}})$ metrizable?

Definition 1.22 [Equivalence] Two metric spaces are **topologically equivalent** if they give rise to the same topology.

Example 1.21 Metrics d_1, d_2, d_∞ in $\mathbb{R}^n$ are topologically equivalent.

1.6.3. Closed Subsets

Definition 1.23 [Closed] Let $(X,\mathcal{T})$ be a topology space. Then $V \subseteq X$ is **closed** if $X \setminus V \in J$ ■

■ **Example 1.22** Under the topology space $(\mathbb{R},\mathcal{T}_{\text{usual}})$, $(b,\infty)\bigcup(-\infty,a) \in \mathcal{T}$. Therefore,

$$[a,b] = \mathbb{R} \setminus \left((b,\infty)\bigcup(-\infty,a)\right)$$

is closed in $\mathbb{R}$ under usual topology. ■

R It is important to say that V is **closed in** X**.** You need to specify the underlying the space X.