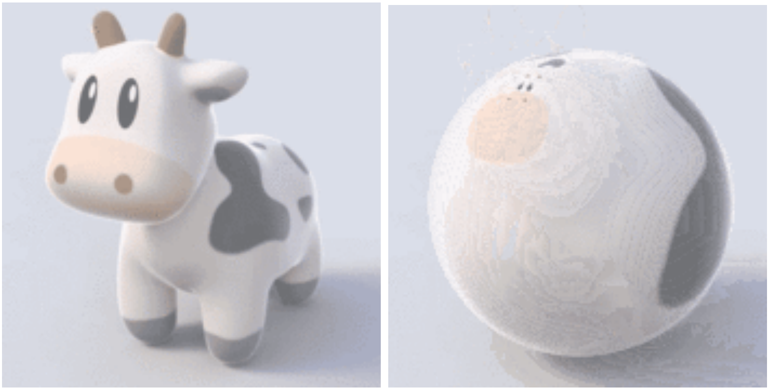
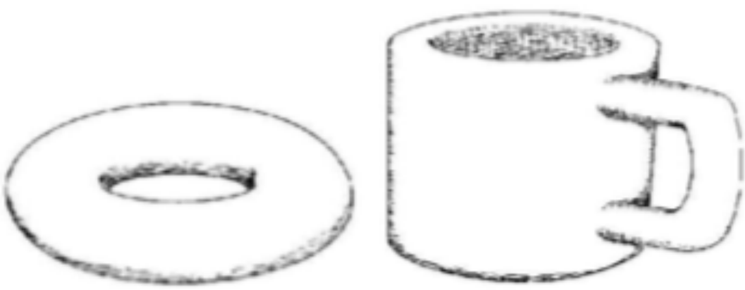
1.3. Monday for MAT4002

1.3.1. Introduction to Topology

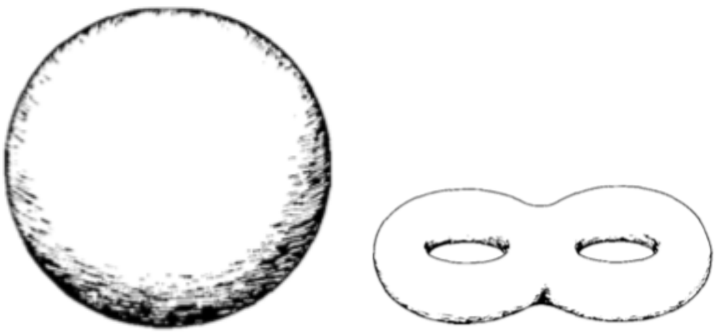
We will study global properties of a geometric object, i.e., *the distrance between 2 points in an object is totally ignored*. For example, the objects shown below are essentially invariant under a certain kind of transformation:



Another example is that the coffee cup and the donut have the same topology:



However, the two objects below have the intrinsically different topologies:



In this course, we will study the phenomenon described above mathematically.

1.3.2. Metric Spaces

In order to ingnore about the distances, we need to learn about distances first.

Definition 1.7 [Metric Space] Metric space is a set X where one can measure distance between any two objects in X.

Specifically speaking, a metric space X is a non-empty set endowed with a function (distance function) $d : X \times X \to \mathbb{R}$ such that

1. $d(\boldsymbol{x}, \boldsymbol{y}) \geq 0$ for $\forall \boldsymbol{x}, \boldsymbol{y} \in X$ with equality iff $\boldsymbol{x} = \boldsymbol{y}$
2. $d(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{y}, \boldsymbol{x})$
3. $d(\boldsymbol{x}, \boldsymbol{z}) \leq d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z})$ (triangular inequality)

■

■ **Example 1.10** 1. Let $X = \mathbb{R}^n$, with

$$d_2(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max_{i=1,\dots,n} |x_i - y_i|$$

2. Let X be any set, and define the discrete metric

$$d(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

Homework: Show that (1) and (2) defines a metric. ■

Definition 1.8 [Open Ball] An **open ball** of radius r centered at $\boldsymbol{x} \in X$ is the set

$$B_r(\boldsymbol{x}) = \{\boldsymbol{y} \in X \mid d(\boldsymbol{x}, \boldsymbol{y}) < r\}$$

■

■ **Example 1.11** 1. The set $B_1(0,0)$ defines an open ball under the metric $(X = \mathbb{R}^2, d_2)$, or the metric $(X = \mathbb{R}^2, d_\infty)$. The corresponding diagram is shown below:

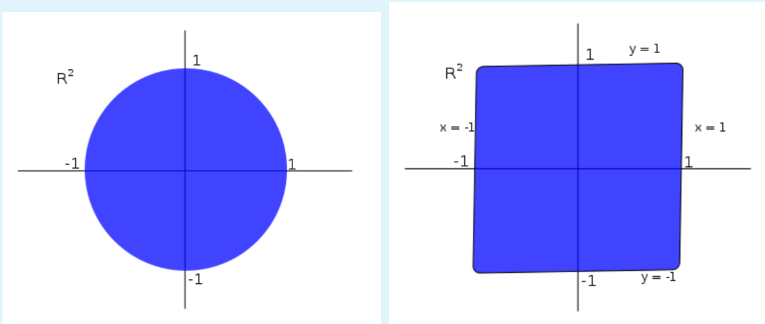


Figure 1.3: Left: under the metric $(X = \mathbb{R}^2, d_2)$; Right: under the metric $(X = \mathbb{R}^2, d_\infty)$

2. Under the metric $(X = \mathbb{R}^2, \text{discrete metric})$, the set $B_1(0,0)$ is one single point, also defines an open ball.

■

Definition 1.9 [Open Set] Let X be a metric space, $U \subseteq X$ is an open set in X if $\forall u \in U$, there exists $\epsilon_u > 0$ such that $B_{\epsilon_u}(u) \subseteq U$. ■

Definition 1.10 The **topology** induced from (X,d) is the collection of all open sets in (X,d), denoted as the symbol $\mathcal{T}$. ■

Proposition 1.5 All open balls $B_r(\boldsymbol{x})$ are open in (X,d).

Proof. Consider the example $X = \mathbb{R}$ with metric d_2. Therefore $B_r(x) = (x-r, x+r)$. Take $\boldsymbol{y} \in B_r(\boldsymbol{x})$ such that $d(\boldsymbol{x},\boldsymbol{y}) = q < r$ and consider $B_{(r-q)/2}(\boldsymbol{y})$: for all $z \in B_{(r-q)/2}(\boldsymbol{y})$, we have

$$d(\boldsymbol{x},\boldsymbol{z}) \le d(\boldsymbol{x},\boldsymbol{y}) + d(\boldsymbol{y},\boldsymbol{z}) < q + \frac{r-q}{2} < r,$$

which implies $\boldsymbol{z} \in B_r(x)$. ■

Proposition 1.6 Let $(X,\boldsymbol{d})$ be a metric space, and $\mathcal{T}$ is the topology induced from (X,d), then

1. let the set $\{G_\alpha \mid \alpha \in \mathcal{A}\}$ be a collection of (uncountable) open sets, i.e., $G_\alpha \in \mathcal{T}$,

then $\bigcup_{\alpha \in \mathcal{A}} G_\alpha \in \mathcal{T}$.

2. let $G_1, \ldots, G_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} G_i \in \mathcal{T}$. The finite intersection of open sets is open.

Proof.

1. Take $x \in \bigcup_{\alpha \in \mathcal{A}} G_\alpha$, then $x \in G_\beta$ for some $\beta \in \mathcal{A}$. Since G_β is open, there exists $\epsilon_x > 0$ s.t.
$$B_{\epsilon_x}(x) \subseteq G_\beta \subseteq \bigcup_{\alpha \in \mathcal{A}} G_\alpha$$
2. Take $x \in \bigcap_{i=1}^{n} G_i$, i.e., $x \in G_i$ for $i = 1, \ldots, n$, i.e., there exists $\epsilon_i > 0$ such that $B_{\epsilon_i}(x) \subseteq G_i$ for $i = 1, \ldots, n$. Take $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$, which implies
$$B_\epsilon(x) \subseteq B_{\epsilon_i}(x) \subseteq G_i, \forall i$$
which implies $B_\epsilon(x) \subseteq \bigcap_{i=1}^{n} G_i$

■

Exercise.

1. let $\mathcal{T}_2, \mathcal{T}_\infty$ be topologies induced from the metrices d_2, d_∞ in $\mathbb{R}^2$. Show that $J_2 = J_\infty$, i.e., every open set in $(\mathbb{R}^2, d_2)$ is open in $(\mathbb{R}^2, d_\infty)$, and every open set in $(\mathbb{R}^2, d_\infty)$ is open in $(\mathbb{R}_2, d_2)$.
2. Let $\mathcal{T}$ be the topology induced from the discrete metric (X, d_{discrete}). What is $\mathcal{T}$?