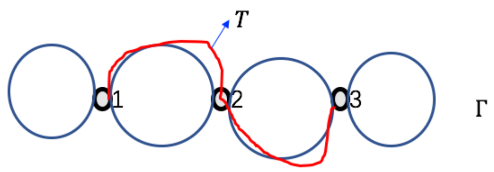
15.3. Monday for MAT4002

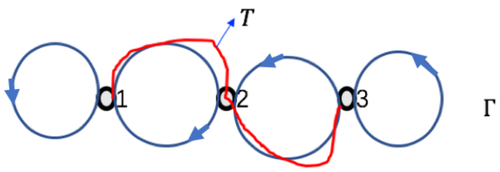
Theorem 15.4 Let Γ be a connected graph. Then $\pi(\Gamma)$ is isomorphic to the free group generated by $\#\{E(\Gamma)\setminus E(T)\}$ elements, for any maximal tree of Γ.

Now we give a proof for this theorem on one special case of Γ:

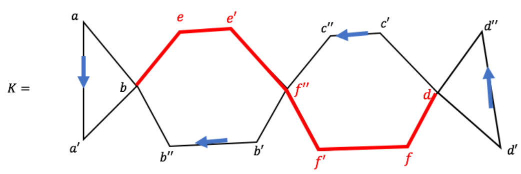


Proof.

- Fix an orientation for each $e \in E(\Gamma)\setminus E(T)$:



- Now let K be a simplicial complex with $|K| \cong \Gamma$:



  As a result, $E(K,b) \cong \pi_1(\Gamma)$

- Now we construct the group homomorphism

$$\begin{aligned} \phi: \quad & \langle \alpha,\beta,\gamma,\delta\rangle \to E(K,b) \\ \text{with} \quad & \phi(\alpha) = [ba'a''b] \\ & \phi(\beta) = [bee'f''b'b''b] \\ & \phi(\gamma) = [bee'f''f'fdc'c''f''e'eb] \\ & \phi(\delta) = [bee'f''f'fdd''d'dff'f''e'eb] \end{aligned}$$

- We can show the group homomorphism ϕ is bijective. In particular, the inverse of ϕ is given by:

$$\Psi: \quad E(K,b) \to \langle \alpha,\beta,\gamma,\delta \rangle$$

where for any $[\ell] := [bv_1 \cdots v_n] \in E(K,b)$, the mapping $\Psi[\ell]$ is constructed by

(a) Remove all other letters appearing in ℓ except $b, a', a'', b', b'', c', c'', d', d''$

(b) Assign

$$\alpha, \ \alpha^{-1}, \ \beta, \ \beta^{-1}, \ \gamma, \ \gamma^{-1}, \ \delta, \ \delta^{-1}$$

for each appearance of

$$a'a'', \ a''a', \ b'b'', \ b''b', \ c'c'', \ c''c', \ d'd'', \ d''d',$$

respectively.

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15.4. The Selfert-Van Kampen Theorem

Theorem 15.5 Let $K = K_1 \cup K_2$ be the union of two **path-connected open** sets, where $K_1 \cap K_2$ is also path-connected. Take $b \in K_1 \cap K_2$, and suppose the group presentations for $\pi_1(K_1,b), \pi_1(K_2,b)$ are

$$\pi_1(K_1,b) \cong \langle X_1 \mid R_1 \rangle, \quad \pi_1(K_2,b) \cong \langle X_2 \mid R_2 \rangle.$$

Let the inclusions be

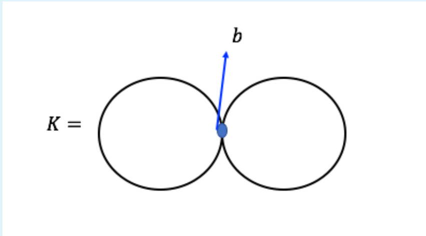
$$i_1 : K_1 \cap K_2 \hookrightarrow K_1, \quad i_2 : K_1 \cap K_2 \hookrightarrow K_2,$$

then a presentation of $\pi_1(K,b)$ is given by:

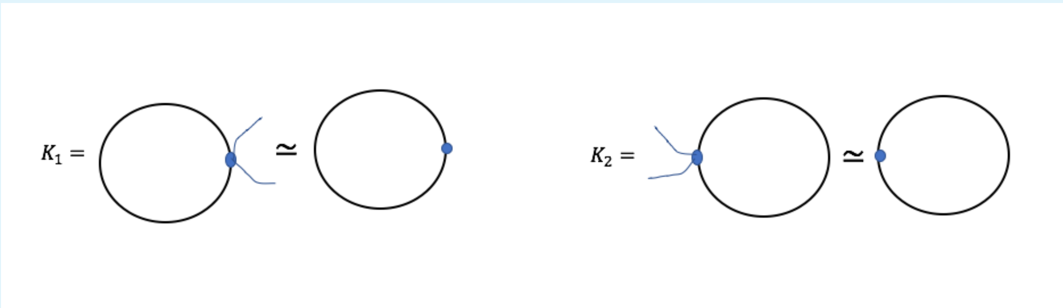
$$\pi_1(K,b) \cong \langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{(i_1)_*(g) = (i_2)_*(g) : \forall g \in \pi_1(K_1 \cap K_2, b)\} \rangle.$$

(Here $(i_1)_* : \pi_1(K_1 \cap K_2, b) \hookrightarrow \pi_1(K_1, b)$ and $(i_2)_* : \pi_1(K_1 \cap K_2, b) \hookrightarrow \pi_1(K_2, b)$.)

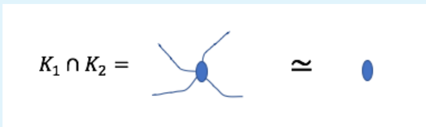
▪ **Example 15.4** 1. Let $K = S^1 \wedge S^1$ given by



(a) Then construct b as the intersection between two circles, and construct K_1, K_2 as shown below:



We can see that $K_1 \cap K_2$ is contractible:



(b) As we have shown before, $\pi_1(S^1) \cong \mathbb{Z}$, which follows that

$$\pi_1(K_1, b) \cong \langle \alpha \rangle, \quad \pi_1(K_2, b) \cong \langle \beta \rangle$$

Also, $\pi_1(K_1 \cap K_2, b) \cong \pi_1(\{b\}, b) \cong \{e\}$.

(c) It's easy to compute $(i_1)_*$ and $(i_2)_*$:

$$\begin{array}{ll} (i_1)_*: & \pi_1(K_1 \cap K_2) \to \pi_1(K_1) \\ \text{with} & e \mapsto e \end{array}, \quad \begin{array}{ll} (i_2)_*: & \pi_1(K_1 \cap K_2) \to \pi_1(K_2) \\ \text{with} & e \mapsto e \end{array},$$

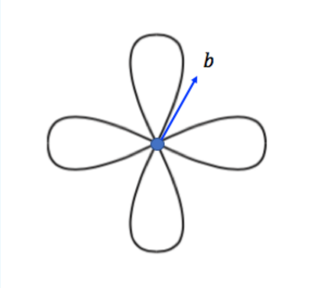
(d) Therefore, by Seifert-Van Kampen Theorem,

$$\pi_1(K,b) \cong \langle \alpha, \beta \mid e = e \rangle \cong \langle \alpha, \beta \rangle$$

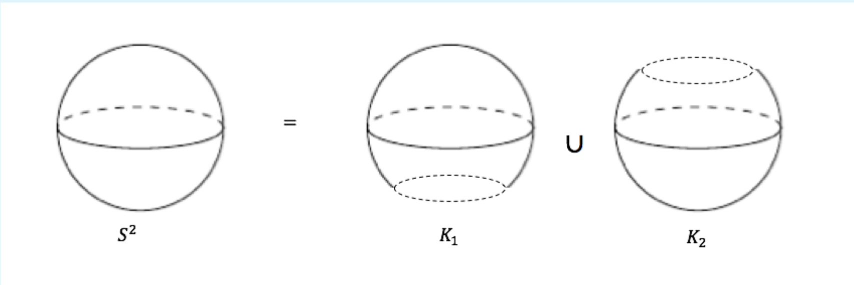
2. By induction,

$$\pi_1(\wedge^n S^1, b) \cong \langle a_1, \ldots, a_n \rangle$$

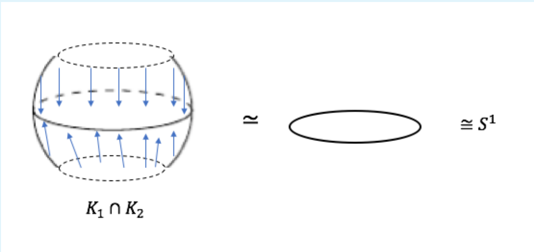
For instance, the figure illustration for $\wedge^4 S^1$ and the basepoint b is given below:



3. (a) Construct $S^2 = K_1 \cup K_2$, which is shown below:



Therefore, we see that $K_1 \cap K_2 \simeq S^1$:



(b) It's clear that K_1 and K_2 are contractible, and therefore

$$\pi_1(K_1) \cong \langle \beta \mid \beta \rangle, \quad \pi_1(K_2) \cong \langle \gamma \mid \gamma \rangle$$

and $\pi_1(K_1 \cap K_2) \cong \pi_1(S^1) \cong \langle \alpha \rangle$.

(c) Then we compute $(i_1)_*$ and $(i_2)_*$. In particular, the mapping $(i_1)_*$ is defined as

$$\begin{aligned} (i_1)_* : \quad & \pi_1(K_1 \cap K_2) \to \pi_1(K_1) \\ \text{with} \quad & [\alpha] \mapsto [i_1(\alpha)] \end{aligned}$$

where α is any loop based at b. Since K_1 is contractible, we imply α in K_1 is homotopic to c_b, i.e.,

$$(i_1)_*([\alpha]) = [i_1(\alpha)] = e, \forall \alpha \in \pi_1(K_1 \cap K_2).$$

Similarly, $(i_2)_*([\alpha]) = e$.

(d) By Seifert-Van Kampen Theorem, we conclude that

$$\pi_1(S^2) \cong \langle \beta, \gamma \mid \beta, \gamma, e = e \rangle \cong \{e\}$$

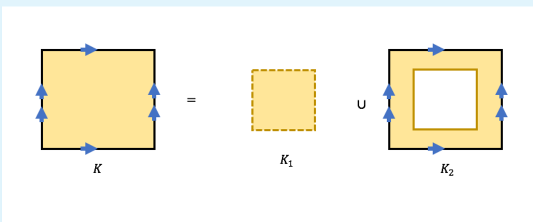
4. Homework: Use the same trick to check that $\pi_1(S^n) = \{e\}$ for all $n \geq 2$. Hint: for S^3, construct

$$K_1 = \{(x_1, \dots, x_4) \in S^3 \mid x_4 > -1/2\}$$

and

$$K_1 = \{(x_1, \dots, x_4) \in S^3 \mid x_4 < 1/2\}$$

5. (a) Consider the quotient space $K \cong \mathbb{T}^2$, and we construct $K = K_1 \cup K_2$ as follows:



Therefore, we can see that K_1 is contractible, and K_2 is homotopy equivalent to $S^1 \wedge S^1$:

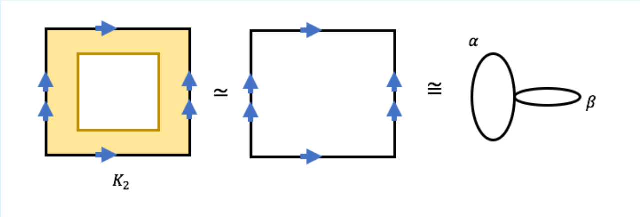
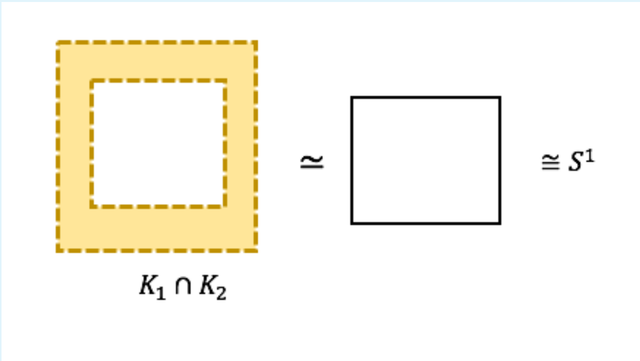


Figure 15.2: Illustration for $K_2 \simeq S^1 \wedge S^1$

and $K_1 \cap K_2$ is homotopic equivalent to the circle:



(b) It follows that

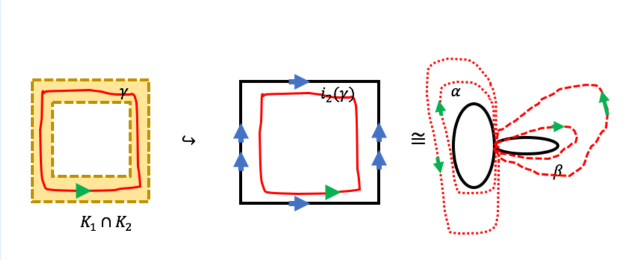
$$\pi_1(K_1) \cong \{e\}, \quad \pi_1(K_2) \cong \langle \alpha, \beta \rangle,$$

and $\pi_1(K_1 \cap K_2) \cong \langle \gamma \rangle$.

(c) Then we compute $(i_1)_*$ and $(i_2)_*$. In particular, $(i_1)_*$ is trivial:

$$\begin{aligned} (i_1)_* : \quad & \pi_1(K_1 \cap K_2) \to \pi_1(K_1) \\ \text{with} \quad & [\alpha] \mapsto e \end{aligned}$$

Then compute $(i_2)_*$. In particular, for any loop γ, we draw the graph for $i_2(\gamma)$:



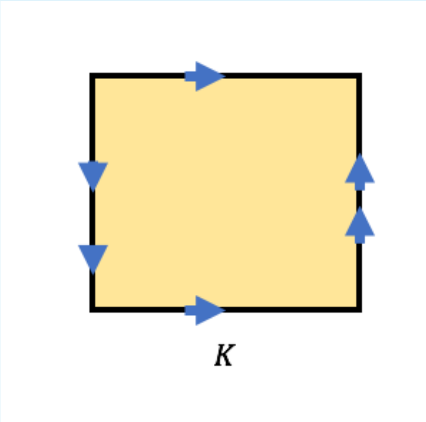
Therefore,

$$(i_2)_*[\gamma] = [i_2(\gamma)] = [\alpha\beta\alpha^{-1}\beta^{-1}]$$

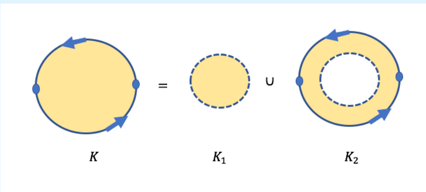
(d) By Seifert-Van Kampen Theorem, we conclude that

$$\pi_1(K) \cong \langle \alpha, \beta \mid \beta, \alpha\beta\alpha^{-1}\beta^{-1} = e \rangle \cong \langle \alpha, \beta \mid, \alpha\beta = \beta\alpha \rangle \cong \mathbb{Z} \times \mathbb{Z}$$

6. Exerise: The Klein bottle K shown in graph below satisfies $\pi_1(K) = \langle a, b \mid aba^{-1}b \rangle$.



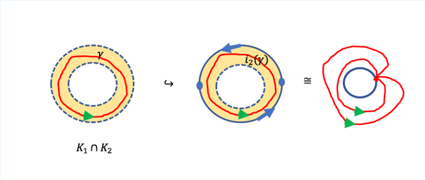
7. Consider the quotient space $K = \mathbb{R}P^2$. We construct $K = K_2 \cup K_2$, which is shown below:



(a) It's clear that K_1 is contractible. In hw3, question 1, we can see that $K_2 \simeq S^1$. Moreover, similar as in (5), $K_1 \cap K_2 \simeq S^1$.

(b) Therefore, $\pi_1(K_1) = \{e\}$ and $\pi_1(K_2) = \langle\alpha\rangle$, $\pi_1(K_1 \cap K_2) = \langle\gamma\rangle$.

(c) It's easy to see that $(i_1)_*([\gamma]) = e$ for any loop γ. For any loop γ, we draw the graph for $i_2(\gamma)$:



Therefore, $(i_2)_*([\gamma]) = [i_2(\gamma)] = [\alpha^2]$.

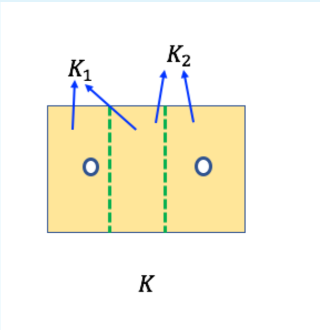
(d) By Seifert-Van Kampen Theorem, we conclude that

$$\pi_1(K) \cong \langle \alpha \mid \alpha^2 = e\rangle \cong \mathbb{Z}/2\mathbb{Z} \cong \{0,1\}_{\text{mod }(2)}$$

8. Let $K = \mathbb{R}^2 \setminus \{2 \text{ points } \alpha, \beta\}$. As have shown in hw3, $K \simeq S^1 \wedge S^1$, which implies

$$\pi_1(K) \cong \pi_1(S^1 \wedge S^1) \cong \langle \alpha, \beta\rangle.$$

We can compute the fundamental group for K directly. Construct $K = K_1 \cup K_2$ as follows:



(a) It's clear that $K_1 \cong \mathbb{R}^2 \setminus \{\text{one point}\} \simeq S^1$ and similarly $K_2 \simeq S^1$. Moreover, $K_1 \cap K_2$ is contractible

(b) Therefore,

$$\pi_1(K_1) \cong \langle\alpha\rangle, \quad \pi_1(K_2) \cong \langle\beta\rangle, \quad \pi_1(K_1 \cap K_2) \cong \{e\}$$

(c) Therefore, $(i_1)_*$ and $(i_2)_*$ is trivial since $\pi_1(K_1 \cap K_2) \cong \{e\}$.

(d) By Seifert-Van Kampen Theorem, we conclude that

$$\pi_1(K) \cong \langle \alpha, \beta \mid e = e \rangle \cong \langle \alpha, \beta \rangle$$

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