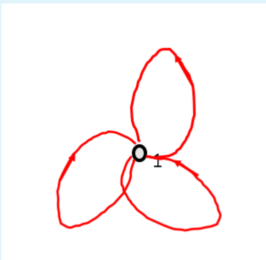
14.3. Monday for MAT4002

14.3.1. Fundamental group of a Graph

Definition 14.3 [Graph] A graph $T = (V, E)$ is defined by the following components:

- V is a finite or countable set, called vertex set;
- E is a finite or countable set, called edge set;
- A function $\delta : E \to V \times V$ with $\delta(e) = (\ell(e), \tau(e))$, where $\ell(e), \tau(e)$ is known as the endpoints of e.

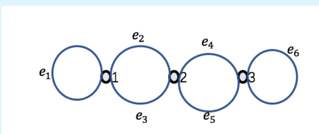
Example 14.2 1. Let $V = \{1\}, E = \{e_1, e_2, e_3\}$, and define $\delta(e_i) = (1,1), i = 1,2,3$. The graph (V, E) is represented below:



2. Let $V = \{e_1, e_2, e_3\}$ and $E = \{e_1, \dots, e_6\}$, and define

$$\delta(e_1) = (1,1), \quad \delta(e_2) = (1,2), \quad \delta(e_3) = (1,2),$$
$$\delta(e_4) = (2,3), \quad \delta(e_5) = (2,3), \quad \delta(e_6) = (3,3).$$

The graph (V, E) is represented below (We do not care the direction of edges for this graph):

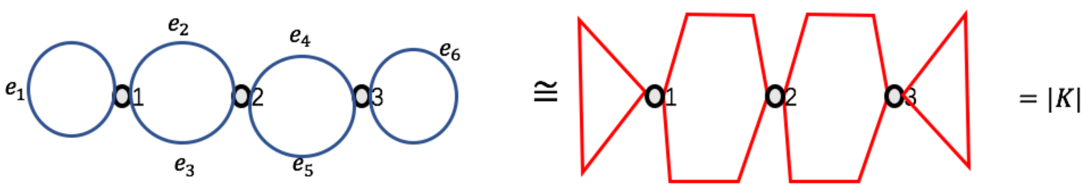


Definition 14.4 [Realizatin of a Graph] For a given graph $\Gamma = (V, E)$, construct a realization by

$$\{|V| \times \{\text{zero simplies}\} \coprod |E| \times \{1\text{-simplies}\}\}/\sim$$

where the equivalence class is induced from the function δ. We still call this realization of the graph as Γ.

(R) In general, graphs are not simplicial complexes. But we can "sub-divide" each edge of Γ into three parts such that there exists simplicial complex K with $|K| \cong \Gamma$. For instance,



where $|K|$ is a simplicial complex.

Definition 14.5

- Subgraph $\Gamma' \subseteq \Gamma$: $\Gamma' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$, and

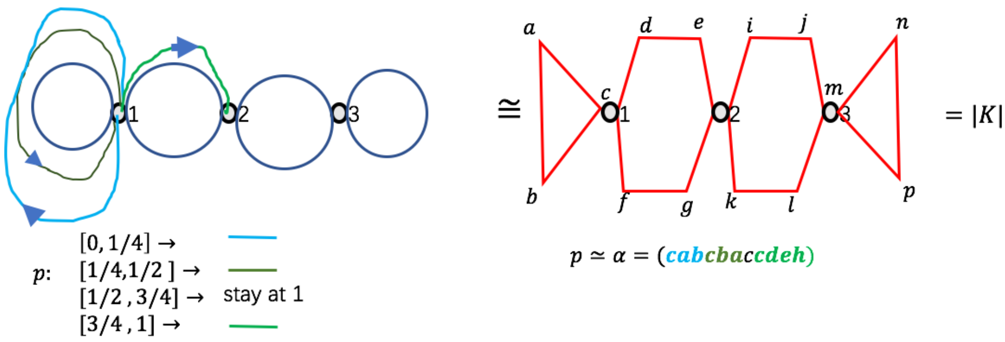
$$\delta\,|_{V'}: E' \to V' \times V'$$

- Edge path: A continous function $p : [0,1] \to \Gamma$ such that there exists $n \in \mathbb{N}$ satisfying

$$p\,|_{[i/n, i+1/n]}: \left[\frac{i}{n}, \frac{i+1}{n}\right] \to T$$

is a path along an edge of Γ, or a constant function on a vertex of Γ, for $0 \le i \le n-1$.

(R) Under the homeomorphism $\Gamma \cong |K|$, each edge path is homotopic to $|g_\alpha|$ for some edge path α in the simplicial complex K. For instance,



- An Edge loop is an edge path p such that $p(0) = p(1) = b \in V$.

- Embedded Edge Loop: An injective edge loop, i.e., $p : [0,1] \to \Gamma$ such that

$$\text{for } x \notin V, \quad p^{-1}(x) = \emptyset \text{ or a single point.}$$

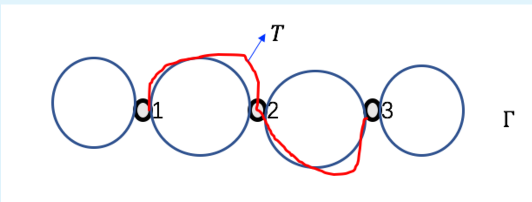
- Tree: a connected graph T that contains no embedded edge loop $p : [0,1] \to T$.

  For instance, as shown in the figure, T_1 contains no edge loop, in particular, the edge loop (a,b,a) is not embedded; T_2 contains embedded edge loop (a,b,c,d,a).

- Maximal Tree of a connected graph Γ:

  - A subgraph T of Γ such that T is a tree.
  - By adding an edge $e \in E(\Gamma) \setminus E(T)$ into T, the new graph is no longer a tree.

  For instance, $T \subseteq \Gamma$ shown in the figure below is a maximal tree.



Theorem 14.5 Let Γ be a connected graph, and T is a subgraph of Γ such that T is a tree. Then T is a maximal tree if and only if $V(T) = V(\Gamma)$.

Moreover, there always exists a maximal tree for all Γ.

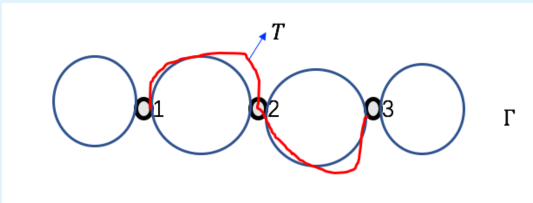
Proof Outline for second part. Construct an ordering of $\{v_1, \ldots, v_i\} \subseteq V(\Gamma)$, such that for each integer $i \geq 2$, there is an edge connecting v_{i+1} with some vertex in $\{v_1, \ldots, v_i\}$.

Then construct $T_1 \subseteq T_2 \subseteq \cdots$, where T_i is a tree containing vertices $\{v_1, \ldots, v_i\}$. As a result, $T = \cup_{i \in \mathbb{N}} T_i$ is a maximal tree. ■

Theorem 14.6 Let Γ be a connected graph. Then $\pi(\Gamma)$ is isomorphic to the free group generated by $\#\{E(\Gamma) \setminus E(T)\}$ elements, for any maximal tree of Γ.

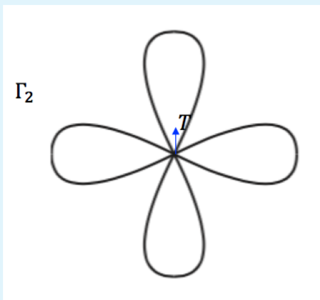
Example 14.3

1. The graph $T \subseteq \Gamma_1$ shown in the figure below is a maximal tree.



Therefore, $\pi_1(\Gamma_1) \cong \langle a, b, c, d \rangle$ since $\#\{E(\Gamma_1) \setminus E(T)\} = 4$.

2. The graph $T \subseteq \Gamma_2$ shown in the figure below is a maximal tree.



Therefore, $\pi_1(\Gamma_2) \cong \langle a, b, c, d \rangle$ since $\#\{E(\Gamma_2) \setminus E(T)\} = 4$.

3. Note that $\Gamma_1 \simeq \Gamma_2$.