13.6. Wednesday for MAT4002

13.6.1. Applications on the isomorphism of fundamental group

Theorem 13.6

$$\pi_1(S^1) \cong (\mathbb{Z}, +)$$

Proof. Define the orientation of $|K|$ as shown in Fig. (13.3).

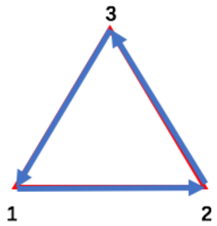

Figure 13.3: Orientation of $|K|$

Following the proof during last lecture, we construct

$$\begin{aligned} \phi: \quad & E(K,1) \to (\mathbb{Z}, +) \\ \text{with} \quad & [\alpha] \mapsto \text{winding number of } \alpha \end{aligned}$$

where the winding number of α is the

$$\text{number of 23 appearing in } \alpha - \text{number of 32 appearing in } \alpha.$$

Note that

1. The winding number is invariant under elementary contraction and elementary expansion.

2. In particular,

$$\text{winding number for } (1\underbrace{23\cdots123}_{23 \text{ shows for } m \text{ times}}1) = m$$
$$\text{winding number for } (1\underbrace{32\cdots132}_{32 \text{ shows for } n \text{ times}}1) = -n$$

3. For any given α, it is equivalent to a unique $(123123\cdots1231)$ or $(132\cdots1321)$, since otherwise α will have different winding numbers.

Therefore, (1) and (3) shows the well-definedness of ϕ. In particular, (1) shows that as $\alpha \sim \alpha'$, we have $\phi([\alpha]) = \phi([\alpha'])$; (2) shows that the winding number of α is an unique integer.

- Homomorphism: For given any two edge loops α, β based at 1, suppose that $[\alpha] = [(1bc1bc\cdots1bc1)]$ and $[\beta] = [(1pq1pq\cdots1pq1)]$, then

$$\phi([\alpha]\cdot[\beta]) = \phi([\alpha\cdot\beta]) = [(1bc1bc\cdots1bc11pq1pq\cdots1pq1)]$$

  Discuss the case for the sign of $\phi([\alpha])$ and $\phi([\beta])$ separately gives the desired result.

- Surjectivity: for a given $m \in \mathbb{Z}$, construct α such that $\phi([\alpha]) = m$ is easy.

- Injectivity: suppose that $\phi([\alpha]) = 0$, then by definition of ϕ, $[\alpha] = [(1)] = e$, which is the trivial element in $E(K, 1)$.

Therefore, ϕ is an isomorphism. ■

R Actually, we can show that the loop based at 1 given by:

$$\begin{aligned} \ell \quad & I \to S^1 \\ \text{with} \quad & t \mapsto e^{2\pi i t} \end{aligned}$$

is a generator for $\pi_1(S^1, 1)$:

- $\phi([\ell]) = 1$, where $\phi : \pi_1(S^1, 1) \cong \mathbb{Z}$.
- The loop

$$\begin{aligned} \ell^m : \quad & I \to S^1 \quad m \in \mathbb{Z} \\ \text{with} \quad & \ell^m(t) = e^{2\pi i m t} \end{aligned}$$

gives $\phi([\ell^m]) = m$

Corollary 13.4 [Fundamental Theorem of Algebra] All non-constant polynomials in $\mathbb{C}$ has at least one root in $\mathbb{C}$

Proof.

- Suppose on the contrary that

$$p(x) = a_n x^n + \cdots + a_1 x + a_0 \; a_n \neq 0$$

has no roots, then p is a mapping from $\mathbb{C}$ to $\mathbb{C} \setminus \{0\}$. It's clear that $\mathbb{C} \setminus \{0\} \simeq \{z \in \mathbb{C} \mid |z| = 1\}$, and therefore

$$\pi_1(\mathbb{C} \setminus \{0\}) = \pi_1(S^1) \cong \mathbb{Z}.$$

- The induced homomorphism p^* of p is given by:

$$\begin{aligned} p_* : \quad & \pi_1(\mathbb{C}) \to \pi_1(\mathbb{C} \setminus \{0\}) \\ \text{with} \quad & \{e\} \mapsto \mathbb{Z} \end{aligned}$$

Note that $\pi_1(\mathbb{C})$ is trival as $\mathbb{C}$ is contractible. Also, $p_*(e) = 0$.

- Consider the inclusion from $C_r = \{z \in \mathbb{C} \mid |z| = r\}$ to $\mathbb{C}$:

$$\begin{aligned} i : \quad & C_r \to \mathbb{C} \\ \text{with} \quad & z \mapsto z \end{aligned}$$

It satisfies the diagram given below:

$$\begin{tikzcd} C_r \arrow[r, dashed, "i"] \arrow[dr, "p|_{C_r}"'] & \mathbb{C} \arrow[d, dashed, "p"] \\ & \mathbb{C} \setminus \{0\} \end{tikzcd}$$

As a result, the induced homomorphism i^* of i satisfies the diagram

$$\begin{tikzcd} \pi_1(C_r) \arrow[r, dashed, "i_*"] \arrow[dr, "\left(p|_{C_r}\right)_*"'] & \pi_1(\mathbb{C}) \arrow[d, dashed, "p_*"] \\ & \pi_1(\mathbb{C} \setminus \{0\}) \end{tikzcd}$$

Or equivalently,

$$\begin{tikzcd} \mathbb{Z} \cong \pi_1(C_r) \arrow[r, dashed, "i_*"] \arrow[dr, "\left(p|_{C_r}\right)_*"'] & \{e\} \arrow[d, dashed, "p_*"] \\ & \mathbb{Z} \cong \pi_1(\mathbb{C} \setminus \{0\}) \end{tikzcd}$$

Therefore, $p_* \circ i_*$ is a zero map since $p_*(e) = 0$, i.e., $(p\,|_{C_r})_*$ is a zero homomorphism.

- Then it's natural to study $p\,|_{C_r}\colon C_r \to \mathbb{C} \setminus \{0\}$. Construct

$$\begin{cases} q(z) = k \cdot z^n, \quad k := \dfrac{p(r)}{r^n} \text{is a constant} \\ p(z) = a_n z^n + \cdots + a_1 z + a_0 \end{cases}$$

Therefore, $p(r) = q(r)$, and $p|_{C_r}, q|_{C_r} : C_r \to \mathbb{C} \setminus \{0\}$.

- We claim that $p|_{C_r} \simeq q|_{C_r}$ for large r. First construct the mapping

$$\begin{aligned} H: \quad & C_r \times [0,1] \to \mathbb{C} \\ \text{with} \quad & H(z,t) = tp(z) + (1-t)q(z) \\ \text{and} \quad & H(z,0) = q(z),\ H(z,1) = p(z) \end{aligned}$$

  If we want to show H is the homotopy between $p|_{C_r}$ and $q|_{C_r}$, it suffices to show that H is well-defined, i.e., $H : c_r \times [0,1] \to \mathbb{C} \setminus \{0\}$.

  Suppose on the contrary that there exists (z,t) such that

$$(1-t)p(z) + tq(z) = 0, \quad |z| = r, t \in [0,1]$$

  Or equivalently,

$$(1-t)(a_n z^n + \cdots + a_1 z + a_0) + t \cdot kz^n = 0.$$

  Substituting k with $p(r)/r^n$ gives

$$a_n z^n + \cdots + a_1 z + a_0 = t\left(a_{n-1} z^{n-1} + \cdots + a_0 - a_{n-1}\frac{z^n}{r} - \cdots - a_1 \frac{z^n}{r^{n-1}} - a_0 \frac{z^n}{r^n}\right)$$

  The LHS has leading order n, while the RHS has leading order less or equal to $n-1$. As $r = |z| \to \infty$, $t \to \infty$. Therefore, the equality does not hold in the range $t \in [0,1]$ when r is sufficiently large.

  For this choice of $r = |z|$,

$$H : c_r \times [0,1] \to \mathbb{C} \setminus \{0\}$$

  gives the homotopy $p|_{C_r} \simeq q|_{C_r}$.

- Therefore, we imply $(p|_{C_r})_* = (q|_{C_r})_*$. Now we check the mapping $(q|_{C_r})_* : \mathbb{Z} \to \mathbb{Z}$. In particular, we check the value of $(q|_{C_r})_*(e)$, where e is the identity in $\pi_1(C_r)$.

Here we construct the loop

$$\begin{aligned} \ell : \quad & I \to C_r \\ \text{with} \quad & \ell(t) = re^{2\pi it} \end{aligned}$$

and therefore $[\ell] = e$. It follows that

$$(q|_{C_r})_*(e) = (q|_{C_r})_*([\ell]) = [q|_{C_r}(\ell)] = q(re^{2\pi it}) = k\cot r^n \cdot e^{2\pi int} \neq 0.$$

Therefore, $(q|_{C_r})_*$ is not a zero homomorphism, i.e., $(p|_{C_r})_* \cong (q|_{C_r})_*$ is not a zero homomorphism, which is a contradiction.

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