12.6. Wednesday for MAT4002

Reviewing.

- Edge loop based at $b \in V$:

$$\alpha = (b, v_1, \cdots, v_n, b)$$

- Equivalence class of edge loops:

$$[\alpha] = \{\alpha' \mid \alpha' \sim \alpha, \alpha' \text{ is the edge loop based at } b\}$$

  Note that $\alpha' \sim \alpha$ if they differ from finitely many elementary contractions or expansions.

  For instance, let K in the figure below denote a triangle:

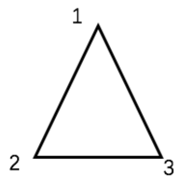


Figure 12.1: Triangle K

  Then the canonical form of any equivalence form $[\alpha]$ can be expressed as:

$$[\alpha] = [bcabc \cdots ab],$$

  where $a, b, c \in \{1, 2, 3\}$ are distinct.

12.6.1. Groups & Simplicial Complices

Proposition 12.6 The $E(K, b) = \{[\alpha] \mid \alpha \text{ is edge loop based at } b\}$ is a group, with the operation

$$[\alpha] * [\beta] = [\alpha \cdot \beta]$$

Proof. 1. Well-definedness of $*$:

$$\alpha \sim \alpha', \beta \sim \beta' \implies \alpha \cdot \beta \sim \alpha' \cdot \beta'$$

2. Associativity is clear.
3. The identity is $e := [b]$: for any edge loop $[\alpha] = [bv_1 \cdots b]$,

$$\begin{aligned}[\alpha] * e &= [bv_1 \cdots v_n b] * [b] \\ &= [bv_1 \cdots v_n bb] \\ &= [bv_1 \cdots v_n b] = [\alpha].\end{aligned}$$

   Also, $e * [\alpha] = [\alpha]$.
4. The inverse of any edge loop $[bv_1 \cdots v_n b]$ is $[bv_n \cdots v_1 b]$:

$$\begin{aligned}[bv_1 \cdots v_n b]^{-1} * [bv_1 \cdots v_n b] &= [bv_n \cdots v_1 bbv_1 \cdots v_n b] \\ &= [bv_n \cdots v_1 bv_1 \cdots v_n b] \\ &= [bv_n \cdots v_2 v_1 v_2 \cdots v_n b] \\ &= \cdots \\ &= [b]\end{aligned}$$

   Similarly, $[bv_1 \cdots v_n b] * [bv_1 \cdots v_n b]^{-1} = [b]$.

∎

We will see that for K defined in Fig.(12.1), $E(K, 1) \cong \mathbb{Z}$, in the next class.

Theorem 12.5 $E(K, b) \cong \pi_1(|K|, b)$.

This is the most difficult theorem that we have faced so far. Let's recall the simplicial approximation proposition first:

Proposition 12.7 — **Simplicial Approximation Proposition.** Suppose that $f : |K| \longrightarrow |L|$ be such that for all $v \in V(K)$, there exists $g(v) \in V(L)$ satisfying

$$f(\mathrm{st}_K(v)) \subseteq \mathrm{st}_L(g(v)).$$

As a result,

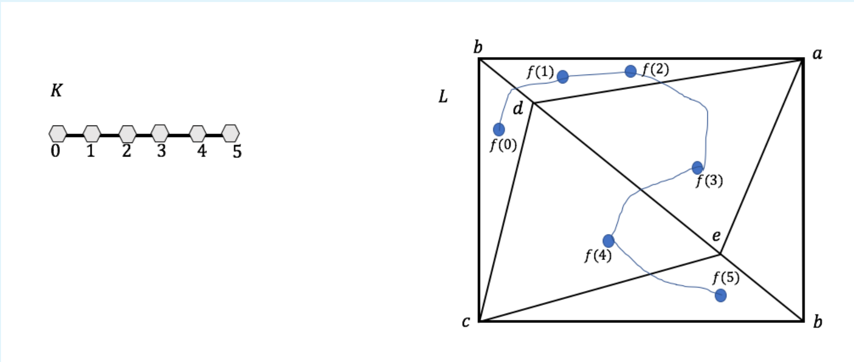
1. the mapping

$$\begin{aligned} g: \quad & K \longrightarrow L \\ \text{with} \quad & v \mapsto g(v) \end{aligned}$$

   is a simplicial map, i.e., for all $\sigma_K \in \Sigma_K, g(\sigma_K) \in \Sigma_L$
2. Moreover, $|g| \simeq f$.

Furthermore, if $A \subseteq K$ is a simplicial subcomplex such that $f(|A|) \subseteq |B|$, where $B \subseteq L$ is a simplicial subcomplex, then we can choose g such that $g\,|_A \colon A \longrightarrow B$ and the homotopy $|g| \simeq f$ sends $|A|$ to $|B|$.

■ **Example 12.9** Consider the simplicial complex K and L shown in the figure below:



Let A_1 denote the subcomplex with $V(A_1) = \{0\}, \Sigma_{A_1} = \emptyset$, and A_2 denote the subcomplex wit $V(A_2) = \{1,2\}$ and $\Sigma_{A_2} = \{\{1,2\}\}$. Therefore,

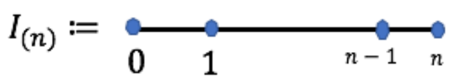
$$f(|A_1|) \subseteq |\Delta_{\{b,c,d\}}|, \quad f(|A_2|) \subseteq |\Delta_{\{a,b,d\}}|,$$

There exists simplicial mapping g with

$$g(0) = b, \quad g(1) = b, \quad g(2) = d, \quad g(3) = e, \quad g(4) = c, \quad g(5) = c$$

∎

Proof. 1. For each edge loop $\alpha = (v_0, \ldots, v_n)$ based at b, consider the simplicial complex



Together with the simplicial map

$$\begin{aligned} g_\alpha : \quad & I_{(n)} \longrightarrow K \\ \text{with} \quad & g_\alpha(i) = v_i \end{aligned}$$

Note that it is well-defined since $\{i, i+1\} \in \Sigma_{I_{(n)}}$, and $\{v_i, v_{i+1}\} \in \Sigma_K$.

Now construct the mapping

$$\begin{aligned} \theta : \quad & \{\text{edge loop based at } b\} \longrightarrow \pi_1(K, b) \\ \text{with} \quad & \alpha \mapsto [|g_\alpha|] \\ \text{where} \quad & |g_\alpha| : |I_{(n)}| (\cong [0,1]) \longrightarrow |K| \\ & |g_\alpha|(i/n) = v_i \end{aligned}$$

For example,

$$\alpha = (bdeabcb), \implies |g_\alpha|(0) = b, |g_\alpha|(1/6) = d, |g_\alpha|(2/6) = e, \cdots, |g_\alpha|(1) = b,$$

i.e., $|g_\alpha|$ is a loop based at b.

Therefore, $[|g_\alpha|] \in \pi_1(|K|, b)$.

2. Now, suppose $\alpha \sim \alpha'$ be two edge loops differ by an elemenary contraction, e.g.,

$$\alpha' = (bdebcb) \sim \alpha = (bdeabcb).$$

As a result, $|g_{\alpha'}| \simeq |g_\alpha|$ relative to $\{0,1\}$, i.e., $[|g_\alpha|] = [|g_{\alpha'}|]$.

Therefore, we have a well-defined map:

$$\begin{aligned} \tilde{\theta}: &\quad \{\text{edge loops based at } b\}/\!\sim \longrightarrow \pi_1(|K|, b) \\ \text{with} &\quad [\alpha] \mapsto [|g_\alpha|] \end{aligned}$$

Therefore, $\tilde{\theta}: E(K,b) \to \pi_1(|K|, b)$ is the desired map.

3. $\tilde{\theta}$ is a homomorphism: it suffices to show that

$$\tilde{\theta}([\alpha] * [\beta]) = \tilde{\theta}([\alpha])\tilde{\theta}([\beta]),$$

which suffices to show $[|g_{\alpha\cdot\beta}|] = [|g_\alpha||g_\beta|]$, i.e., $|g_{\alpha\cdot\beta}| \simeq |g_\alpha||g_\beta|$. Note that $|g_{\alpha\cdot\beta}|$ and $|g_\alpha||g_\beta|$ are the same path with different "speed", i.e., homotopy.

4. The mapping $\tilde{\theta}$ is surjective: Let $\ell : [0,1] \to |K|$ be a loop based at b. It suffices to find an edge loop α such that $[|g_\alpha|] = [\ell]$, i.e., $|g_\alpha| \simeq \ell$.

   (a) Applying the simplicial approximation theorem, there exist large n and $g : I_{(n)} \to K$ such that $|g| \simeq \ell$. Here we can choose $g : I_{(n)} \to K$ to be such that $g(\{0\}) = \{b\}$, $g(\{n\}) = \{b\}$, and $|g| \simeq \ell$ relative to $\{0,n\}$.

   (b) Take $\alpha = (g(0), g(1), \ldots, g(n))$ so that $g(0) = b = g(n)$, with $g_\alpha = g$. Therefore, $[|g_\alpha|] = [\ell]$, and hence $\tilde{\theta}$ is surjective.

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