12.3. Monday for MAT4002

Proposition 12.3 If b, b' are path connected in X, then

$$\pi_1(X,b) \cong \pi_1(X,b')$$

R Last lecture we have given the isomorphism

$$\begin{aligned} W_\# : \quad & \pi_1(X,b) \to \pi_1(X,b') \\ \text{with} \quad & [\ell] \mapsto [w^{-1} \cdot \ell \cdot w] \end{aligned}$$

where w denotes a path from b to b'. The inverse of $W_\#$ is given by:

$$\begin{aligned} W_\#^{-1} : \quad & \pi_1(X,b') \to \pi_1(X,b) \\ \text{with} \quad & [m] \mapsto [w \cdot m \cdot w^{-1}] \end{aligned}$$

Notation. For path connected space X, we will just write $\pi_1(X)$ instead of $\pi_1(X,x)$.

Proposition 12.4 Let (X,x) and (Y,y) be spaces with basepoints x and y, and $f: X \to Y$ be a continuous map with $f(x) = y$. Then every loop $\ell : I \to X$ based at x gives a loop $f \circ \ell : I \to Y$ based at y, i.e., the continous map f induces a homomorphism of groups

$$\begin{aligned} f_* : \quad & \pi_1(\pi, x) \to \pi_1(Y,y) \\ & [\ell] \mapsto [f \circ \ell] := f_*([\ell]) \end{aligned}$$

Moreover,

1. $(\mathrm{id}_{X\to X})_* = \mathrm{id}_{\pi_1(X,x)\to\pi_1(X,x)}$
2. $(g \circ f)_* = g_* \circ f_*$
3. If $f \simeq f'$ relative to $x \in X$, then $f_* = (f')_*$

Proof. • Well-definedness: Suppose that $\ell \simeq \ell'$, then $f \circ \ell \simeq f \circ \ell'$ by propositon (9.4). Therefore, $[f \circ \ell] = [f \circ \ell']$.

- Homomorphism: It's clear that

$$f \circ (\ell \circ \ell') = (f \circ \ell) \circ (f \circ \ell')$$

  Therefore, $f_*[\ell\ell'] = (f_*[\ell]) * (f_*[\ell'])$

The other three statements are obvious.

■

Proposition 12.5 Let X, Y be path-connected such that $X \simeq Y$ (i.e., there exists $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X, f \circ g \simeq \mathrm{id}_Y$). Then $\pi_1(X) \cong \pi_1(Y)$.

In particular, if X, Y are path-connected with $X \cong Y$, then $\pi_1(X) \cong \pi_1(Y)$

Proof. Consider the mapping

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1)$$

It suffices to show that f_* and g_* are bijective. (The homomorphism follows from proposition (12.4))

- Wrong proof: $g \circ f \simeq \mathrm{id}_X$ implies $(g \circ f)_* = (\mathrm{id}_X)_*$ implies $g_* \circ f_* = \mathrm{id}_{\pi_1(X, x_0)}$.

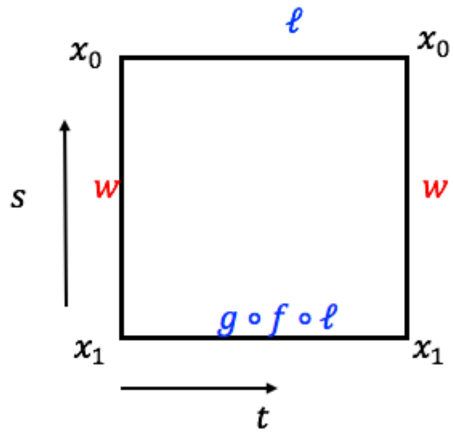
  Reason: note that $(g \circ f) \simeq \mathrm{id}_X$ is **not** relative to x_0.

Consider the homotopy $H : g \circ f \simeq \mathrm{id}_X$, where $H(x_0, s)$ is not necessarily a constant for $s \in I$. It follows that $H(x_0, 0) = x_1$ and $H(x_0, 1) = x_0$, i.e., $w(s) := H(x_0, s)$ defines a path from x_1 to x_0.

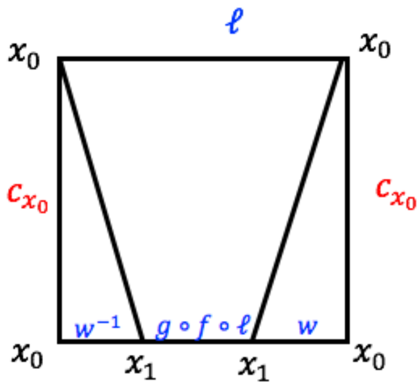
For any loop $\ell : I \to X$ based at x_0, consider the homotopy

$$\begin{aligned}
K = H \circ (\ell \times \mathrm{id}_I) : \quad & I \times I \to X \\
\text{where} \quad & K(t, s) = H((\ell(t), s)) \\
& K(t, 0) = H(\ell(t), 0) = g \circ f(\ell(t)) \\
& K(t, 1) = H(\ell(t), 1) = \ell(t) \\
& K(0, s) = w(s) = K(1, s)
\end{aligned}$$

The graphic plot of K is given in the figure below:



The homotopy between ℓ and $g \circ f \circ \ell$ motivates us to construct a homotopy between ℓ and $w^{-1} \circ g \circ f \circ \ell \circ w$ relative to $\{0,1\}$:



Therefore,

$$[\ell] = [w^{-1} g f \ell w] = W_\#([g f \ell]) = (W_\# \circ g_* \circ f_*)[\ell]$$

which follows that $W_\# \circ g_* \circ f_* = \mathrm{id}_{\pi_1(X,x_0)}$. Therfore, f_* is injective, g_* is surjective.

The similar argument gives

$$W_\# \circ f_* \circ g_* = \mathrm{id}_{\pi_1(Y,y_0)}$$

Therefore, f_* is surjective, g_* is injective. The bijectivity is shown.

∎

Definition 12.1 [Simply-Connected] A space X is **simply-connected** if X is path connected, and X has trivial fundamental group, i.e., $\pi_1(X) = \{e\}$ for some point $e \in X$. ▪

▪ **Example 12.4** If X is contractible, then X is path-connected. By proposition (12.5), since $X \simeq \{e\}$, we imply

$$\pi_1(X) \cong \pi_1(\{e\}) = \{e\}.$$

Therefore, all contractible spaces (e.g., $\mathbb{R}^n$) are simply-connected.

However, not all simply-connected spaces are contractible, e.g., $\pi_1(S^2) \cong \{e\}$, but S^2 is not homotopy equivalent to a point. ▪

12.3.1. Some basic results on $\pi_1(X, b)$

We will study $\pi_1(X, b)$ for some simplicial complexes.

Definition 12.2 [Edge Loop] Let $K = (V, \Sigma)$ be a simplicial complex.

1. An edge path $(v_0, \dots, v_n)$ is such that
   (a) $a_i \in V(K)$
   (b) For each i, $\{a_{i-1}, a_i\}$ spans a simplex of K
2. An edge loop is an edge path with $a_n = a_0$.
3. Let $\alpha = (v_0, \dots, v_n), \beta = (w_0, \dots, w_m)$ be two edge paths with $v_n = w_0$, then we define

$$\alpha \circ \beta = (v_0, \dots, v_n, w_1, \dots, w_m)$$

▪

Definition 12.3 [Elementary Contraction/Expansion] Let α, β be two edge paths.

1. An elementary contraction of α is a new edge path obtained by performing one of the followings on α:

- Replacing $\cdots a_{i-1}a_i\cdots$ by $\cdots a_{i-1}\cdots$ provided that $a_{i-1}=a_i$
- Replacing $\cdots a_{i-1}a_ia_{i+1}\cdots$ by $\cdots a_{i-1}\cdots$ provided that $a_{i-1}=a_{i+1}$
- Replacing $\cdots a_{i-1}a_ia_{i+1}\cdots$ by $\cdots a_{i-1}a_{i+1}\cdots$ provided that $\{a_{i-1},a_i,a_{i+1}\}$ spans a 2-simplex of K.

2. An elementary expansion is the reverse of the elementary contraction.
3. Two edge paths α,β are equivalent if α and β differs by a finite sequence of elementary contractions or expansions.