11.6. Wednesday for MAT4002

11.6.1. The fundamental group

Revewing. One example for Homotopy relative to $\{0,1\}$ is illustrated in Fig.(11.4)

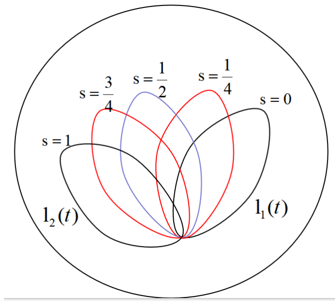


Figure 11.5: Example of homotopy relative to $\{0,1\}$

It's **essential** to study homotopy relative to $\{0,1\}$. For example, given a torus with a loop $\ell_1(t)$ and a base point b. We want to distinguish $\ell_1(t)$ and $\ell_2(t)$ as shown in Fig.(11.6):

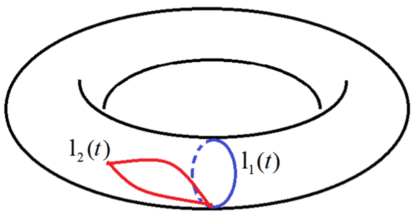


Figure 11.6: Two loops on a torus

Obviously there should be something different between $\ell_1(t)$ and $\ell_2(t)$. "Relative to$\{0,1\}$ is essential", sicne if we get rid of this condition, all loops are homotopic to the constant map $c_b(t) = b$. See the graphic illustration in Fig.(**??**):

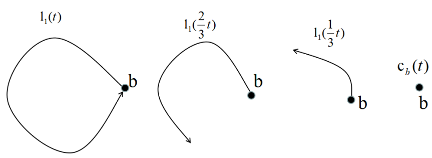


Figure 11.7: homotopy between any loop and constant map

In this case, $\ell \simeq c_b$ for any loop ℓ, there is only one trivial element $\{[c_b]\}$ in $\pi_1(X,b)$.

That's the reason why we define $\pi_1(X,b)$ as the collection of homotopy classes **relative to** $\{0,1\}$ based at b in X.

Proposition 11.13 Let $[\cdot]$ denote the homotopy class of loops relative to $\{0,1\}$ based at b, and define the operation

$$[\ell] * [\ell'] = [\ell \cdot \ell']$$

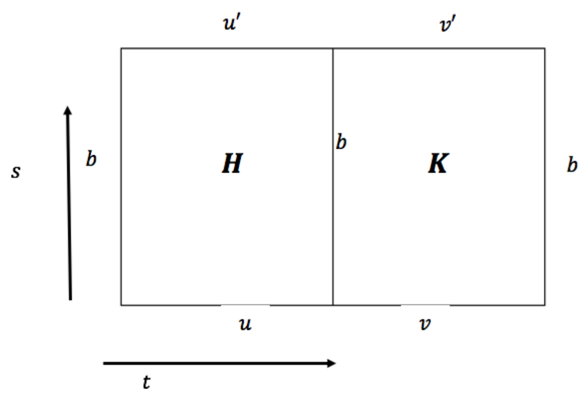
Then $(\pi_1(X,b), *)$ forms a group, where

$$\pi_1(X,b) := \{[\ell] \mid \ell : [0,1] \to X \text{ denotes loops based at } b\}$$

Proof. 1. Well-definedness: Suppose that $u \sim u'$ and $v \sim v'$, it suffices to show $u \cdot v \simeq u' \cdot v'$. Consider the given homotopies $H : u \simeq u'$, $K : v \simeq v'$. Construct a new homotopy $L : I \times I \to X$ by

$$L(t,s) = \begin{cases} H(2t,s), & 0 \le t \le 1/2 \\ K(2t-1,s), & 1/2 \le t \le 1 \end{cases}$$

The diagram below explains the ideas for constructing L. The plane denote the set $I \times I$, and the labels characterize the images of each point of $I \times I$ under L.

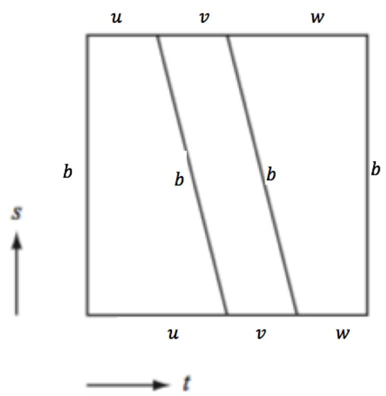


Therefore, $u \cdot v \simeq u' \cdot v'$.

2. Associate: $(u \cdot v) \cdot w \simeq u \cdot (v \cdot w)$

   Note that $(u \cdot v) \cdot w$ and $u \cdot (v \cdot w)$ are essentially different loops. Although they go with the same path, they are with different speeds. Generally speaking, the loop $(u \cdot v) \cdot w$ travels u, v using $1/4$ seconds, and w in $1/2$ seconds; but the loop $u \cdot (v \cdot w)$ travels u in $1/2$ seconds, and then v, w in $1/4$ seconds.

   We want to construct a homotopy that describes the loop changes from $u \cdot (v \cdot w)$ to $(u \cdot v) \cdot w$. A graphic illustration is given below:



   An explicit homotopy $H : I \times I \to X$ is given below:

$$H(t,s) = \begin{cases} u(4t/(2-s)), & 0 \le t \le 1/2 - 1/4s \\ v(4t - 2 + s), & 1/2 - 1/4s \le t \le 3/4 - 1/4s \\ w(4t - 3 + s/(1+s)), & 3/4 - 1/4s \le t \le 1 \end{cases}$$

   Therefore,

$$[u] * ([v] * [w]) = ([u] * [v]) * [w]$$

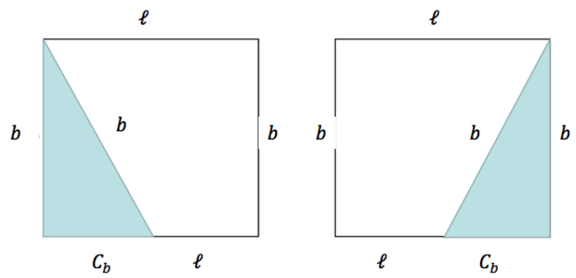
3. Intuitively, the identity should be the constant map, i.e., let $c_b : I \to X$ by $c_b(t) = b, \forall t$, and let $\ell = [c_b]$, it suffices to show

$$[c_b] * [\ell] = [\ell] * [c_b] = [\ell] \iff [c_b \cdot \ell] = [\ell \cdot c_b] = [\ell]$$

   Or equivalently,

$$c_b \cdot \ell \simeq \ell, \quad \ell \cdot c_b \simeq \ell$$

The graphic homotopy is shown below. (You should have been understood this diagram)



4. Inverse: the inverse of $[u]$, where u is a loop, should be $[u']$, where u' is the reverse of the traveling of u. Therefore, for all $u : I \to X$ (loop based at b), define $u^{-1} : I \to X$ by $u^{-1}(t) = u(1-t)$. Note that

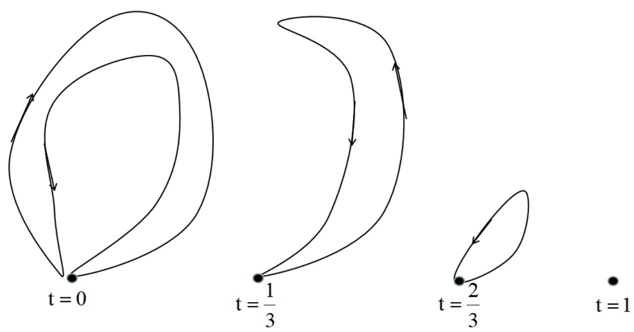
$$[u] * [u^{-1}] = [u \cdot u^{-1}], \quad e = [c_b]$$

It suffices to show $u \cdot u^{-1} \simeq c_b$ and $u^{-1} \cdot u \simeq c_b$:

The homotopy below gives $u \cdot u^{-1} \simeq c_b$, and the $u^{-1} \cdot u \simeq c_b$ follows similarly.

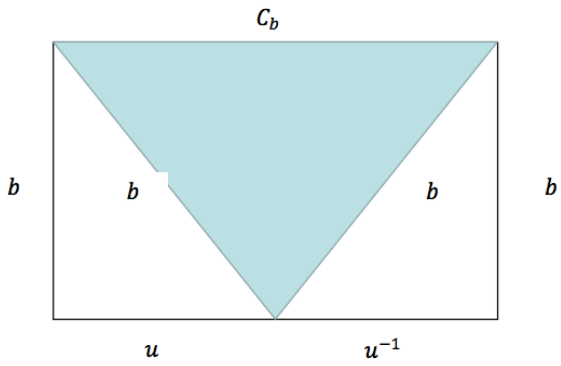
$$H(t,s) = \begin{cases} u(2t(1-s)), & 0 \le t \le 1/2 \\ u((2-2t)(1-s)), & 1/2 \le t \le 1 \end{cases}$$

The graphic illustration is given below:



∎

Ⓡ Note that the figure below does not define a homotopy from $u \cdot u^{-1}$ to c_b!



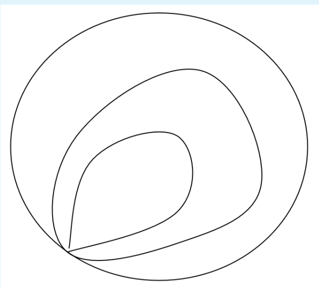
The reason is that for the upper part, as $s \to 1$, the time for traveling u and u^{-1} becomes very small, i.e., a particle has to pass u and u^{-1} in infinitely small time, which is not well-defined.

■ **Example 11.11** The reason why $\pi_1(\mathbb{R}^2, b) = \{e\}$ is trivial:

- For any $u : I \to \mathbb{R}^2$ with $u(0) = u(1) = b$, consider the homotopy

$$H(t,s) = (1-s)u(t) + sb.$$

Therefore, $u \simeq c_b$ for any loop u based at b. Check the diagram below for graphic illustration of this homotopy.



More generally, if $X \simeq \{x\}$ is contractible, then $\pi_1(X, b) = \{e\}$.

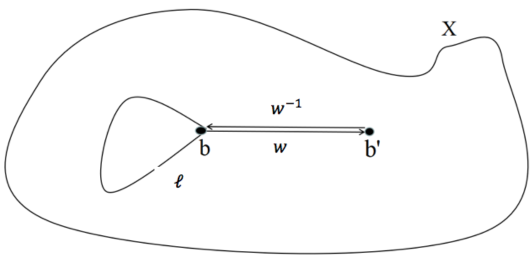
However, $\pi_1(S^1, 1)$ is not trivial. We cannot deform the loop in S^1 into a constant loop. We will see that In fact, $\pi_1(S^1, 1) \cong \mathbb{Z}$. ▪

Proposition 11.14 If b, b' are path-connected in X, then $\pi_1(X, b) \cong \pi_1(X, b')$.

Proof. Let w be a path from b to b', and define

$$\begin{aligned} w_\# : \quad & \pi_1(X, b) \to \pi_1(X, b') \\ \text{with} \quad & [\ell] \mapsto [w^{-1}\ell w] \end{aligned}$$

1. Well-definedness: Check that $\ell \simeq \ell'$ implies $w^{-1}\ell w \simeq w^{-1}\ell' w$. See the figure below for graphic illustration.



2. $w_\#$ is a homomorphism:

$$w_\#([\ell_1]) \cdot w_\#([\ell_2]) = [w^{-1} \cdot \ell_1 w] \cdot [w^{-1} \cdot \ell_2 w] \tag{11.4a}$$
$$= [w^{-1} \cdot \ell_1 \ell_2 w] \tag{11.4b}$$
$$= w_\#([\ell_1 \ell_2]) \tag{11.4c}$$

   where (11.4b) is because that $w \cdot w^{-1} = c_b$.

3. And $w_\#$ is also injective. If loops ℓ_1, ℓ_2 are such that $w_\#(\ell_1) = w_\#(\ell_2)$, then

$$[w^{-1}\ell_1 w] = [w^{-1}\ell_2 w],$$

which follows that

$$[\ell_1] = [w][w^{-1}\ell_1 w][w^{-1}] = [w][w^{-1}\ell_2 w][w^{-1}] = [\ell_2] \tag{11.5}$$

4. Finally, $w_\#$ is surjective, because for any $u \in \pi_1(X, b')$, let $v = wuw^{-1}$, then v is based at b, so $[v] \in \pi_1(X, b)$, and $w_\#(v) = [u]$. Therefore $w_\#$ is surjective.

In conclusion, $w_\#$ is a group isomorphism between $\pi_1(X, b)$ and $\pi_2(X, b)$. ■

R In (11.5) we extended the meaning of $[\ell]$ to allow ℓ to be a path, and the equivalence class is defined by the relation "$\sim$": $\ell_1 \sim \ell_2$ iff they are homotopic relative to $\{0,1\}$. The multiplication rules are defined similarly.