11.3. Monday for MAT4002

Reviewing. Consider the group with presentation $\langle S \mid R(S)\rangle$.

1. The elements in S are generators that have studied in abstract algebra
2. The "relations" of this group are given by the equalities on hte right-hand side, e.g., the dihedral group is defined as

$$\langle a,b \mid a^n = e, b^2 = e, bab = a^{-1}\rangle$$

Sometimes we also simplify the equality $\times = e$ as $\times$, e.g., the dihedral group can be re-written as

$$\langle a,b \mid a^n, b^2, bab = a^{-1}\rangle$$

Example 11.4 Consider

$$G =< a,b \mid a^2, b^2, abab^{-1}a^{-1}b^{-1} >:=< a,b \mid a^2, b^2, aba = bab >= \{e,a,b,ab,ba,aba\}$$

It's isomorphic to S^3, and the shape of S^3 is illustrated in Fig.(11.1)

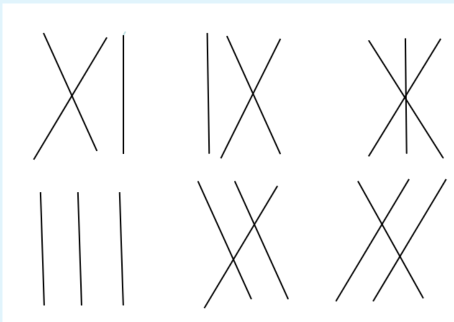

Figure 11.1: Illustration of group S^3

More precisely, the isomorphism is given by:

$$\begin{aligned} \phi: \quad & S_3 \to G \\ \text{with} \quad & X| \mapsto a, \quad |X \mapsto b \end{aligned}$$

■ **Example 11.5** Consider $G_2 =< a,b \mid ab = ba >$ and any word, which can be expressed as $\cdots a^s b^t a^u b^v \cdots$

- If $s \in \mathbb{N}$, we write $a^s := \underbrace{a \cdots a}_{s \text{ times}}$
- If $s \in -\mathbb{N}$, we write $a^s := \underbrace{(a^{-1}) \cdots (a^{-1})}_{-s \text{ times}}$
- For the word with the form $a \cdots b \cdots ba \cdots a$, we can always push a into the leftmost using the relation $ab = ba$
- For the word with the form $a \cdots ab \cdots ba^{-1}$, we can always push a^{-1} into the leftmost using the relation $ba^{-1} = a^{-1}b$.

Therefore, all elements in G_2 are of the form $a^p b^q, p, q \in \mathbb{Z}$, and we have the relation

$$(a^{p_1} b^{q_1})(a^{p_2} b^{q_2}) = a^{p_1+p_2} b^{q_1+q_2}.$$

Therefore, $G_2 \cong \mathbb{Z} \times \mathbb{Z}$, where the isomorphism is given by:

$$\begin{aligned} \phi : \quad & \mathbb{Z} \times \mathbb{Z} \to G_2 \\ \text{with} \quad & (p,q) \mapsto a^p b^q \end{aligned}$$

■

■ **Example 11.6**

$$G_3 =< a \mid a^5 >= \{1, a, a^2, \dots, a^4\}$$

It's clear that $G_3 \cong \mathbb{Z}/5\mathbb{Z}$, where the isomorphism is given by:

$$\begin{aligned} \phi : \quad & \mathbb{Z}/5\mathbb{Z} \to G_3 \\ \text{with} \quad & m + 5\mathbb{Z} \mapsto a^m \end{aligned}$$

■

11.3.1. Cayley Graph for finitely presented groups

Graphs have strong connection with groups. Here we introduce a way of building graphs using groups, and the graphs are known as Cayley graphs. They describe many properties of the group in a topological way.

Definition 11.5 [Oriented Graph] An oriented graph T is specified by

1. A countable or finite set V, known as vertices
2. A countable or finite set E, known as edges
3. A function $\delta : E \to V \times V$ given by

$$\delta(e) = (\ell(e), \tau(e))$$

where $\ell(e)$ denotes the initial vertex and $\tau(e)$ denotes the terminal vertex.

For example, let

- $V = \{a, b, c\}$
- $E = \{e_1, e_2, e_3, e_4\}$
- $\delta(e_1) = (a, a), \delta(e_2) = (b, c), \delta(e_3) = (a, c), \delta(e_4) = (b, c)$

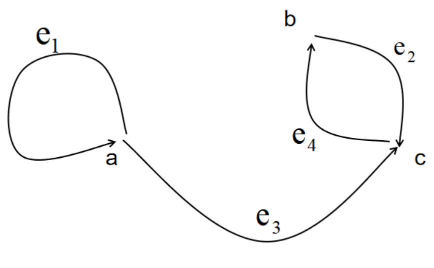

Figure 11.2: Illustration of example oriented graph

The resulted graph is plotted in Fig.(11.2)

Definition 11.6 [Cayley graph] Let $G = \langle S \mid R(S) \rangle$ with $|S| < \infty$. The **Cayley graph** associated to G is an oriented graph with

1. The vertex set G
2. The edge set $E := G \times S$
3. The function $\ell : E \to V \times V$ is given by:

$$\begin{aligned} \ell: \quad & G \times S \to G \times G \\ \text{with} \quad & (g,s) \mapsto (g, g \cdot s) \end{aligned}$$

In particular, we link two elements in G if they differ by a generator rightside.

Example 11.7 1. The Cayley graph for $G = \langle a \rangle$ ($\cong \mathbb{Z}$) is shown in Fig.(11.3):

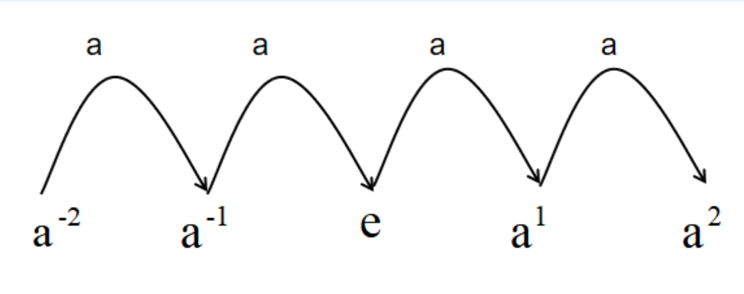

Figure 11.3: Illustration of Cayley Graph $\langle a \rangle$

2. The Cayley graph for $G = \langle a \mid a^3 \rangle$ is shown in Fig.(11.4):

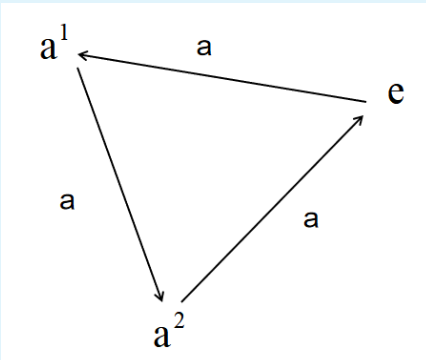

Figure 11.4: Illustration of Cayley Graph $\langle a \mid a^3 \rangle$

3. The Cayley graph for $G = \langle a,b \mid a^2, b^2, aba = bab \rangle$ is shown in Fig.(11.5):

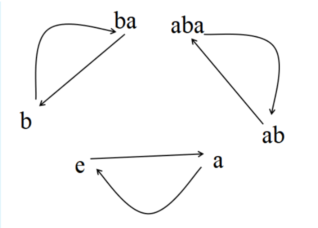

Figure 11.5: Illustration of Cayley Graph $\langle a, b \mid a^2, b^2, aba = bab \rangle$

4. The Cayley graph for $G = \langle a, b \mid ab = ba \rangle$ is shown in Fig.(11.7):

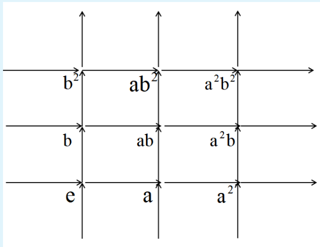

Figure 11.6: Illustration of Cayley Graph $\langle a, b \mid ab = ba \rangle$

5. The Cayley graph for $G = \langle a, b \rangle$ is shown in Fig.(11.8):

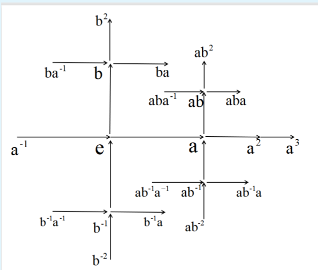

Figure 11.7: Illustration of Cayley Graph $\langle a, b \mid ab = ba \rangle$

(R) There could be different presentations $\langle S_1 \mid R(S_1) \rangle \cong \langle S_2 \mid R(S_2) \rangle$ of the same group.

11.3.2. Fundamental Group

Motivation. The fundamental group connects topology and algebra together, by labelling a group to each topological space, which is known as fundamental group.

Why do we need algebra in topology. Consider the S^2 (2-shpere) and $S^1 \times S^1$ (torus):

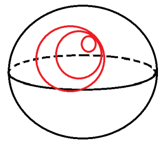

Figure 11.8: Any loop in the sphere can be contracted into a point

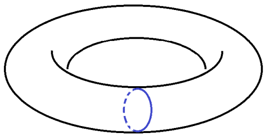

Figure 11.9: Some loops in the torus cannot be contracted into a point

As can be seen from Fig.(11.8) and Fig.(11.9), any "loop" on a sphere can be contracted to a point, while some "loop" on a torus cannot. We need the algebra to describe this phenomena formally.

Definition 11.7 [loop] Let X be a topological space. A **loop** on X is a constant map $\ell : [0,1] \to X$ such that $\ell(0) = \ell(1)$.

We say ℓ is based at $b \in X$ if $\ell(0) = \ell(1) = b$.

Definition 11.8 [composite loop] Suppose that $\boldsymbol{u}, \boldsymbol{v}$ are loops on X based at $b \in X$. The **composite loop** $u \cdot v$ is given by

$$u \cdot v = \begin{cases} u(2t), & \text{if } 0 \le t \le 1/2 \\ v(2t-1), & \text{if } 1/2 \le t \le 1 \end{cases}$$

Definition 11.9 [fundamental group] The **homotopy class of loops relative to** $\{0,1\}$ **based at** $b \in X$ forms a group. It is called the **fundamental group** of X based at b, denoted as $\pi_1(X,b)$.

More precisely, let

$$[\ell] = \{m \mid m \text{ is a loop based at } b \text{ that is homotopic to } \ell, \text{ relative to } \{0,1\}\},$$

and $\pi_1(X,b) = \{[\ell] \mid \ell \text{ are loops based at } b\}$. The operation in $\pi_1(X,b)$ is defined as:

$$[\ell] * [\ell'] := [\ell \cdot \ell'], \quad \forall [\ell],[\ell'] \in \pi_1(X,b).$$

R Two paths $\ell_1, \ell_2 : [0,1] \to X$ are homotopic relative to $\{0,1\}$ if we can find $H : [0,1] \times [0,1] \to X$ such that

$$H(t,0) = \ell_1(t), \quad H(t,1) = \ell_2(t)$$

and

$$H(0,s) = \ell_1(0) = \ell_2(0),\ \forall 0 \le s \le 1, \quad H(1,s) = \ell_1(1) = \ell_2(1),\ \forall 0 \le s \le 1$$

Counter example for homotopy but not relative to $\{0,1\}$:

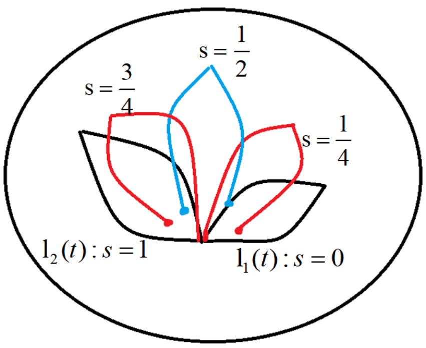

Figure 11.10: homotopy not relative to $\{0,1\}$