Chapter 9

Week9

9.1. Monday for MAT3040

Reviewing.

- $\mathcal{X}_T(x) = (x - \lambda_1)\cdots(x - \lambda_n)$ over $\mathbb{F}$ if and only if T is triangularizable over $\mathbb{F}$.
- $m_T(x) = (x - \mu_1)\cdots(x - \mu_k)$, where μ_i's are distinct over $\mathbb{F}$ if and only if T is diagonalizable over $\mathbb{F}$.

  The converse for this statement is the proposition (8.2). Let's focus on the proof for the forward direction.

9.1.1. Remarks on Primary Decomposition Theorem

Theorem 9.1 — **Primary Decomposition Theorem.** Let $T : V \to V$ be a linear operator with $\dim(V) < \infty$, and

$$m_T(x) = [p_1(x)]^{e_1} \cdots [p_k(x)]^{e_k}$$

where p_i's are distinct, monic, irreducible polynomials. Let $V_i = \ker(p_i(T)^{e_i})$, then

1. each V_i is T-invariant (*i.e.*, $T(V_i) \le V_i$)
2. $V = V_1 \oplus \cdots \oplus V_k$
3. $T\mid_{V_i}$ has the minimal polynomial $p_i(x)^{e_i}$.

Proof. 1. (1) follows from part (2) for example (**??**).

2. Let $q_i(x) = [p_1(x)]^{e_1} \cdots \widehat{[p_i(x)]^{e_i}} \cdots [p_k(x)]^{e_k} := m_T(x)/[p_i(x)]^{e_i}$, then it is clear that

   (a) $\gcd(q_1, \ldots, q_k) = 1$

   (b) $\gcd(q_i, p_i^{e_i}) = 1$

   (c) $q_i \cdot p_i^{e_i} = m_T$

   (d) If $i \neq j$, then $m_T(x) \mid q_i(x)q_j(x)$

   - By (a) and Rezout's Theorem (**??**), there exists polynomials $a_1, \ldots, a_k$ such that

     $$a_1(x)q_1(x) + \cdots + a_k(x)q_k(x) = 1,$$

     which implies

     $$\underbrace{a_1(T)q_1(T)\boldsymbol{v}}_{\boldsymbol{v}_1} + \cdots + \underbrace{a_k(T)q_k(T)\boldsymbol{v}}_{\boldsymbol{v}_k} = \boldsymbol{v}$$

     Therefore, $\boldsymbol{v} = \boldsymbol{v}_1 + \cdots + \boldsymbol{v}_k$ for our constructed $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$.

   - Note that

     $$p_i(T)^{e_i}\boldsymbol{v}_i = p_i(T)^{e_i}a_i(T)q_i(T)\boldsymbol{v} = a_i(T)[q_i(T)p_i(T)^{e_i}]\boldsymbol{v} = a_i(T)m_T(T)\boldsymbol{v} = \boldsymbol{0},$$

     whcih implies $\boldsymbol{v}_i \in \ker([p_i(T)]^{e_i}) := V_i$, and therefore

     $$V = V_1 + \cdots + V_k \tag{9.1}$$

   - To show that the summation in (9.1) is essentially the direct sum, consider

     $$\boldsymbol{0} = \boldsymbol{v}'_1 + \cdots + \boldsymbol{v}'_k, \quad \forall \boldsymbol{v}'_i \in V_i. \tag{9.2}$$

     By (a) and Rezout's Theorem (**??**), there exists $b_i(x), c_i(x)$ such that

     $$b_i(x)q_i(x) + c_i(x)p_i(x)^{e_i} = 1 \implies b_i(T)q_i(T) + c_i(T)p_i(T)^{e_i} = I,$$

     i.e.,

     $$b_i(T)q_i(T)\boldsymbol{v}'_i + c_i(T)p_i(T)^{e_i}\boldsymbol{v}'_i = b_i(T)q_i(T)\boldsymbol{v}'_i = \boldsymbol{v}'_i.$$

Appying the mapping $b_i(T)q_i(T)$ into equality (9.2) both sides, $i = 1, \dots, k$, we obtain

$$\mathbf{0} = b_i(T)q_i(T)\mathbf{0} = b_i(T)q_i(T)\boldsymbol{v}_1' + \cdots + b_i(T)q_i(T)\boldsymbol{v}_k'$$

Note that all terms on RHS vanish except for $b_i(T)q_i(T)\boldsymbol{v}_i' = \boldsymbol{v}_i'$, since $q_i(x) = [p_1(x)]^{e_1} \cdots \widehat{[p_i(x)]^{e_i}} \cdots [p_k(x)]^{e_k}$ and $\boldsymbol{v}_j' \in \ker([p_j(x)]^{e_j})$. Therefore, $\boldsymbol{v}_i' = 0$ for $i = 1, \dots, k$, i.e., $V = V_1 \oplus \cdots \oplus V_k$.

3. For any $\boldsymbol{v}_i \in V_i$, we have $p_i(T)^{e_i}\boldsymbol{v}_i = \mathbf{0}$, which implies $m_{T|V_i}(x) \mid p_i(x)^{e_i}$. Together with Corollary (8.1), $m_{T|v_i}(x) = p_i(x)^{f_i}$ for some $1 \le f_i \le e_i$.
   Suppose on the contrary that there exists $f_i < e_i$ for some i, consider any $\boldsymbol{v} := \boldsymbol{v}_1 + \cdots + \boldsymbol{v}_k \in V$, and

$$p_1(T)^{f_1} \cdots p_k(T)^{f_k}\boldsymbol{v} = p_1(T)^{f_1} \cdots p_k(T)^{f_k}(\boldsymbol{v}_1 + \cdots + \boldsymbol{v}_k)$$

   The term on the RHS vanishes since $p_j(T)^{f_j}\boldsymbol{v}_j = \mathbf{0}$, which implies

$$m_T \mid p_1^{f_1} \cdots p_k^{f_k},$$

   but there exists i such that $e_i > f_i$, which is a contradiction.

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Corollary 9.1 If $m_i(x) = (x - \mu_1) \cdots (x - \mu_k)$ over $\mathbb{F}$, where μ_i's are distinct, then T is diagonalizable over $\mathbb{F}$. (the converse actually also holds, see proposition (8.2))

Proof. By primary decomposition theorem,

$$V = \underbrace{\ker(T - \mu_1 I)}_{V_1} \oplus \cdots \oplus \underbrace{\ker(T - \mu_k I)}_{V_k}$$

Take B_i as a basis of V_i, an μ_i-eigenspace of T. Then $B := \cup_{i=1}^k B_i$ is a basis consisting of eigenvectors of T.

It's clear that $(T|_{V_i})_{\mathcal{B},\mathcal{B}} = \text{diag}(\mu_i, \dots, \mu_i)$, and T is diagonalizable with

$$(T)_{\mathcal{B},\mathcal{B}} = \text{diag}((T|_{V_1})_{\mathcal{B},\mathcal{B}}, \cdots, (T|_{V_k})_{\mathcal{B},\mathcal{B}}).$$

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Corollary 9.2 [Spectral Decomposition] Suppose $T : V \to V$ is diagonalizable, then there exists a linear operator $p_i : V \to V$ for $1 \le i \le k$ such that

- $p_i^2 = p_i$ (idempotent)
- $p_i p_j = 0, \forall i \ne j$
- $\sum_{i=1}^k p_i = I$
- $p_i T = T p_i, \forall i$

and scalars $\mu_1, \dots, \mu_k$ such that

$$T = \mu_1 p_1 + \cdots + \mu_k p_k$$

Proof. Diagonlization of T is equivalent to say that $m_T(x) = (x - \mu_1) \cdots (x - \mu_k)$, where μ_i's are distinct. Construct

- $V_i := \ker(T - \mu_i I)$
- $p_i : V \to V$ given by $p_i = a_i(T)q_i(T)$ as in the proof of primary decomposition theorem

Then:

- $p_i T = T p_i$ is obvious
- $\sum_{i=1}^k p_i = \sum_{i=1}^k a_i(T)q_i(T) = I$
- $p_i p_j = a_i(T)a_j(T)q_i(T)q_j(T) := a_i(T)a_j(T)s(T)m_T(T) = \mathbf{0}$
- $p_i^2 = p_i(p_1 + \cdots + p_k) = p_i \cdot I = p_i$

For the last part, note that

- $p_i V \leq V_i, \forall i$: for $\forall \boldsymbol{v} \in V$,

$$(T - \mu_i I) p_i \boldsymbol{v} = (T - \mu_i I) a_i(T) q_i(T) \boldsymbol{v} = a_i(T) m_T(x) \boldsymbol{v} = \mathbf{0}$$

  Therefore, $p_i V \leq \ker(T - \mu_i I) = V_i$

- Now, for all $\boldsymbol{w} \in V$,

$$\begin{aligned} T\boldsymbol{w} &= T(p_1 + \cdots + p_k)\boldsymbol{w} \\ &= Tp_1\boldsymbol{w} + \cdots + Tp_k\boldsymbol{w} \\ &= (\mu_1 p_1)\boldsymbol{w} + \cdots + (\mu_k p_k)\boldsymbol{w} \end{aligned}$$

  and therefore $T = \mu_1 p_1 + \cdots + \mu_k p_k$

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Organization of future two weeks. We are interested in under which condition does the T is diagonalizable. One special case is $T = A$, where $\boldsymbol{A}$ is a symmetric matrix. We will study normal operators, which includes the case for symmetric matrices.

Question: what happens if $m_T(x)$ contains repeated linear factors? We will spend the next whole class to show the Jordan Normal Form:

Theorem 9.2 — Jordan Normal Form. Let $\mathbb{F}$ be algebraically closed field such that every linear operator $T : V \to V$ has the form

$$m_T(x) = \prod_{i=1}^{k} (x - \lambda_i)^{e_i}$$

where λ_i's are distinct.

Then there exists basis $\mathcal{A}$ of V such that

$$(T)_{\mathcal{A},\mathcal{A}} = \text{diag}(\boldsymbol{J}_1, \ldots, \boldsymbol{J}_k)$$

where

$$\boldsymbol{J}_i = \begin{pmatrix} \mu & 1 & 0 & 0 \\ 0 & \mu & 1 & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu \end{pmatrix}$$

for some $\mu \in \{\lambda_1, \ldots, \lambda_k\}$