Chapter 7

Week7

7.1. Monday for MAT3040

Reviewing. Define the characteristic polynomial for an linear operator T:

$$\mathcal{X}_T(x) = \det((T)_{\mathcal{A},\mathcal{A}} - x\boldsymbol{I})$$

We will use the notation "$I/\boldsymbol{I}$" in two different occasions:

1. I denotes the identity transformation from V to V with $I(\boldsymbol{v}) = \boldsymbol{v}, \forall \boldsymbol{v} \in V$
2. $\boldsymbol{I}$ denotes the identity matrix $(I)_{\mathcal{A},\mathcal{A}}$, defined based on any basis $\mathcal{A}$.

7.1.1. Minimal Polynomial

Definition 7.1 [Linear Operator Induced From Polynomial] Let $f(x) := a_m x^m + \cdots + a_0$ be a polynomial in $\mathbb{F}[x]$, and $T : V \to V$ be a linear operator. Then the mapping

$$f(T) = a_m T^m + \cdots + a_1 T + a_0 I : \quad V \to V,$$

is called a linear operator induced from the polynomial $f(x)$.

Definition 7.2 [Minimal Polynomial] Let $T : V \to V$ be a linear operator. The **minimal polynomial** $m_T(x)$ is a **nonzero monic polynomial** of least (minimal) degree such that

$$m_T(T) = \mathbf{0}_{V \to V}.$$

where $\mathbf{0}_{V\to V}$ denotes the zero vector in $\text{Hom}(V,V)$. ■

■ **Example 7.1** 1. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $\boldsymbol{A}$ defines a linear operator:

$$\begin{aligned} A: \quad & \mathbb{F}^2 \to \mathbb{F}^2 \\ \text{with} \quad & \boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x} \end{aligned}$$

Here $\mathcal{X}_A(x) = (x-1)^2$ and $\boldsymbol{A} - \boldsymbol{I} = \boldsymbol{0}$, which gives $m_A(x) = x - 1$.

2. Let $\boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which implies

$$\mathcal{X}_B(x) = (x-1)^2,$$

The question is that can we get the minimal polynomial with degree 1?

The answer is no, since $\boldsymbol{B} - k\boldsymbol{I} = \begin{pmatrix} 1-k & 1 \\ 0 & 1-k \end{pmatrix} \neq \boldsymbol{0}$.

In fact, $m_B(x) = (x-1)^2$, since

$$(\boldsymbol{B} - \boldsymbol{I})^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

■

Two questions naturally arises:

1. Does $m_T(x)$ exist? If exists, is it unique?
2. What's the relationship between $m_T(x)$ and $\mathcal{X}_T(x)$?

Regarding to the first question, the minimal polynomial $m_T(x)$ may not exist, if V has infinite dimension:

■ **Example 7.2** Consider $V = \mathbb{R}[x]$ and the mapping

$$\begin{aligned} T: \quad & V \to V \\ & p(x) \mapsto \int_0^x p(t)\,\mathrm{d}t \end{aligned}$$

In particular, $T(x^n) = \frac{1}{n+1}x^{n+1}$. Suppose $m_T(x)$ is with degree n, i.e.,

$$m_T(x) = x^n + \cdots + a_1 x + a_0,$$

then

$$m_T(T) = T^n + \cdots + a_0 I \text{ is a zero linear transformation}$$

It follows that

$$[m_T(T)](x) = \frac{1}{n!}x^n + a_{n-1}\frac{1}{(n-1)!}x^{n-1} + \cdots + a_1 x + a_0 = 0_{\mathbb{F}},$$

which is a contradiction since the coefficients of x^k is nonzero on LHS for $k = 1, \ldots, n$, but zero on the RHS. ■

Proposition 7.1 The minimal polynomial $m_T(x)$ always exists for $\dim(V) = n < \infty$.

Proof. It's clear that $\{I, T, \ldots, T^n, T^{n+1}, \cdots, T^{n^2}\} \subseteq \mathrm{Hom}(V, V)$. Since $\dim(\mathrm{Hom}(V, V)) = n^2$, we imply $\{I, T, \ldots, T^n, T^{n+1}, \cdots, T^{n^2}\}$ is linearly dependent, i.e., there exists a_i's that are not all zero such that

$$a_0 I + a_1 T + \cdots + a_{n^2} T^{n^2} = 0$$

i.e., there is a polynomial $g(x)$ of degree less than n^2 such that $g(T) = 0$.

The proof is complete. ■

Proposition 7.2 The minimal polynomial $m_T(x)$, if exists, then it exists uniquely.

Proof. Suppose f_1, f_2 are two distinct minimal polynomials with $\deg(f_1) = \deg(f_2)$. It follows that

- $\deg(f_1 - f_2) < \deg(f_1)$.
- $f_1 - f_2 \neq 0$
- $(f_1 - f_2)(T) = f_1(T) - f_2(T) = 0_{V \to V}$

By scaling $f_1 - f_2$, there is a monic polynomial g with lower degree satisfying $g(T) = 0$, which contradicts the definition for minimal polynomial. ■

Proposition 7.3 Suppose $f(x) \in \mathbb{F}[x]$ satisfying $f(T) = \mathbf{0}$, then

$$m_T(x) \mid f(x).$$

Proof. It's clear that $\deg(f) \geq \deg(m_T)$. The division algorithm gives

$$f(x) = q(x)m_T(x) + r(x).$$

Therefore, for any $\boldsymbol{v} \in V$

$$[r(T)](\boldsymbol{v}) = [f(T)](\boldsymbol{v}) - [q(T)m_T(T)](\boldsymbol{v}) = \mathbf{0}_V - q(T)\mathbf{0}_V = \mathbf{0}_V - \mathbf{0}_V = \mathbf{0}_V$$

Therefore, $r(T) = \mathbf{0}_{V \to V}$. By definition of minimal polynomial, we imply $r(x) \equiv 0$. ■

Proposition 7.4 If $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{F}^{n \times n}$ are similar to each other, then $m_A(x) = m_B(x)$.

Proof. Suppose that $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$, and that

$$m_A(x) = x^k + \cdots + a_1 x + a_0, \quad m_B(x) = x^\ell + \cdots + b_0.$$

It follows that

$$\begin{aligned} m_A(\boldsymbol{B}) &= \boldsymbol{B}^k + \cdots + a_0 I \\ &= \boldsymbol{P}^{-1}\boldsymbol{A}^k\boldsymbol{P} + \cdots + a_0\boldsymbol{P}^{-1}\boldsymbol{P} \\ &= \boldsymbol{P}^{-1}(\boldsymbol{A}^k + \cdots + a_0\boldsymbol{I})\boldsymbol{P} \\ &= \boldsymbol{P}^{-1}(m_A(\boldsymbol{A}))\boldsymbol{P} \end{aligned}$$

Therefore, $m_A(\boldsymbol{B}) = \mathbf{0}$ since $m_A(\boldsymbol{A}) = \mathbf{0}$. By proposition (7.3), we imply $m_B(x) \mid m_A(x)$. Similarly, $m_A(x) \mid m_B(x)$. Since $m_A(x)$ and $m_B(x)$ are monic, we imply $m_A(x) = m_B(x)$. ■

R Proposition (7.4) claims that the minimal polynomial is **similarity-invariant**; actually, the characteristic polynomial is **similarity-invariant** as well.

Assumption. We will asssume V has finite dimension from now on. Now we study the vanishing of a single vector $\boldsymbol{v} \in V$.

Notation. The $m_T(x)$ is a nonzero monic poylnomial of least degree such that

$$m_T(T) = \mathbf{0}_{V \to V}.$$

7.1.2. Minimal Polynomial of a vector

Definition 7.3 [Minimal Polynomial of a vector] Similar to the minimal polynomial, we define the **minimal polynomial of a vector $\boldsymbol{v}$ relative to** T, say $m_{T,\boldsymbol{v}}(x)$, as the monic polynomial of least degree such that

$$m_{T,\boldsymbol{v}}(T)(\boldsymbol{v}) = 0$$

The existence of minimal polynomial of a vector is due to the existence of minimal polynomial; the uniqueness follows similarly as in proposition (7.2).

Proposition 7.5 Let $T : V \to V$ be a linear operator and $\boldsymbol{v} \in V$. The degree of the minimal polynomial of a vector is upper bounded by:

$$\deg(m_{T,\boldsymbol{v}}(x)) \leq \dim(V).$$

Proof. It's clear that $\{\boldsymbol{v}, T\boldsymbol{v}, \ldots, T^n\boldsymbol{v}\} \subseteq V$ and the proof follows similarly as in proposition (7.1). ■

Similar to the division property in proposition (7.3), we have the division proprty for minimal polynomial of a vector:

Proposition 7.6 Suppose $f(x) \in \mathbb{F}[x]$ satisfying $f(T)(\boldsymbol{v}) = \mathbf{0}_V$, then

$$m_{T,\boldsymbol{v}}(x) \mid f(x).$$

In particular, $m_{T,\boldsymbol{v}} \mid m_T(x)$.

Proof. The proof follows similarly as in proposition (7.3). ∎

Proposition 7.7 Suppose that $m_{T,\boldsymbol{v}}(x) = f_1(x)f_2(x)$, where f_1, f_2 are both monic. Let $\boldsymbol{w} = f_1(T)\boldsymbol{v}$, then

$$m_{T,\boldsymbol{w}}(x) = f_2(x)$$

Proof.

1. $$f_2(T)\boldsymbol{w} = f_2(T)f_1(T)\boldsymbol{v} = m_{T,\boldsymbol{v}}(T)\boldsymbol{v} = \mathbf{0}$$

   By the proposition (7.3), we imply $m_{T,\boldsymbol{w}} | f_2$.

2. On the other hand,

   $$\mathbf{0} = m_{T,\boldsymbol{w}}(T)(\boldsymbol{w}) = m_{T,\boldsymbol{w}}(T)f_1(T)\boldsymbol{v} = f_1(T)m_{T,\boldsymbol{w}}(T)\boldsymbol{v},$$

   which implies that $m_{T,\boldsymbol{v}}(x) \mid f_1(x)m_{T,\boldsymbol{w}}(x)$,, i.e.,

   $$f_1 \cdot f_2 \mid f_1 \cdot m_{T,\boldsymbol{w}} \implies f_2 \mid m_{T,\boldsymbol{w}}.$$

   The proof is complete.

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