Chapter 6

Week6

6.1. Monday for MAT3040

6.1.1. Polynomials

We recall some useful properties of polynomial before studying eigenvalues/eigenvectors.

Definition 6.1 [Polynomial]

1. A polynomial over $\mathbb{F}$ has the form

$$p(z) = a_m z^m + \cdots + a_1 z + a_0, \quad (a_m \neq 0).$$

   Here $a_m z^m$ is called the **leading term** of $p(z)$; m is called the degree; a_m is called the **leading coefficient**; $a_m, \cdots, a_0$ are called the coefficients of this polynomial.

2. A polynomial over $\mathbb{F}$ is monic if its leading coefficient is $1_{\mathbb{F}}$.
3. A polynomial $p(z) \in \mathbb{F}[z]$ is **irreducible** if for any $a(z), b(z) \in \mathbb{F}[z]$,

$$p(z) = a(z)b(z) \implies \text{either } a(z) \text{ or } b(z) \text{ is a constant polynomial.}$$

   Otherwise $p(z)$ is **reducible**.

Example 6.1 For example, the polynomial $p(x) = x^2 + 1$ is irreducible over $\mathbb{R}$; but $p(x) = (x-i)(x+i)$ is **reducible** over $\mathbb{C}$.

Theorem 6.1 — Division Theorem. For all $p,q \in \mathbb{F}[z]$ such that $p \neq 0$, there exists unique $s,r \in \mathbb{F}[x]$ satisfying $\deg(r) < \deg(f)$, such that

$$p(z) = s(z) \cdot q(z) + r(z).$$

Here $r(z)$ is called the **remainder**.

Example 6.2 Given $p(x) = x^4 + 1$ and $q(x) = x^2 + 1$, the junior school knowledge tells us that uniquely

$$x^4 + 1 = (x^2 - 1)(x^2 + 1) + 2.$$

Theorem 6.2 — Root Theorem. For $p(x) \in \mathbb{F}[x]$, and $\lambda \in \mathbb{F}$, $x - \lambda$ divides p if and only if $p(\lambda) = 0$.

Proof.

1. If $(x-\lambda)$ divides p, then $p = (x-\lambda)q$ for some $q \in \mathbb{F}[x]$. Thus clearly $p(\lambda) = 0$.
2. For the other direction, suppose that $p(\lambda) = 0$. By division theorem, there exists $s,r \in \mathbb{F}[x]$ such that

$$p = (x-\lambda)s + r \quad \text{with } \deg(r) < \deg(x-\lambda) = 1. \tag{6.1}$$

Therefore, the polynomial r must be constant.

Substituting λ into x both sides in (6.1), we have

$$0 = p(\lambda) = 0 \cdot s + r \implies r = 0.$$

Therefore, $p = (x-\lambda) \cdot s$, i.e., $(x-\lambda)$ divides p.