5.4. Wednesday for MAT3040

There will be a quiz on next Monday.

$$\text{Scope}: \quad \text{From Week 1 up to (including) the definition of } B^*.$$

Reviewing.

1. If V is finite dimensional, and B a basis of V, then B^* is a basis of the dual space V^*.
2. Define the Annihilator $\text{Ann}(S) \le V^*$:

$$\text{Ann}(S) = \{f \in V^* \mid f(s) = 0, \forall s \in S\}$$

3. If V is finite dimensional, and $W \le V$, then $\text{Ann}(W)$ fills the gap, i.e.,

$$\dim(\text{Ann}(W)) = \dim(V) - \dim(W)$$

4. Define a map

$$\begin{aligned} \Phi: \quad & \text{Ann}(W) \to (V/W)^* \\ & f \mapsto \tilde{f} \end{aligned}$$

where $\tilde{f}$ is defined such that the diagram (5.1) below commutes

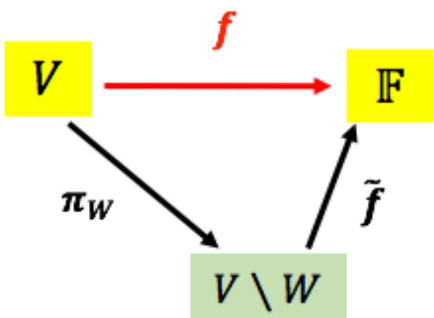

Figure 5.1: Construction of $\tilde{f}$

Or equivalently, $\tilde{f}: V/W \to \mathbb{F}$ is such that $\tilde{f}(\boldsymbol{v} + W) = f(\boldsymbol{v})$.

5.4.1. Adjoint Map

The natural question is that whether Φ is the isomorphism between $\mathrm{Ann}(W)$ and $(V/W)^*$:

Proposition 5.4 Φ is a linear transformation, i.e.,

$$\Phi(af+bg) = a\cdot\Phi(f) + b\cdot\Phi(g).$$

Proof. Itt suffices to show that

$$\overline{af+bg} = a\overline{f} + b\overline{g}$$

■

Therefore, we need to answer whether Φ a bijective map. We will show this conjucture at the end of this lecture. The definition of Φ is **natural**, i.e., we do not need to specify any basis to define this Φ . However, as studied in Monday, the constructed isomorphism $V \to V^*$ with $\boldsymbol{v}_i \mapsto f_i$ is not natural.

Definition 5.3 [Adjoint Map] Let $T: V \to W$ be a linear transformation. Define the **adjoint** of T by

$$T^*: \quad W^* \to V^*$$

such that for any $f \in W^*$,

$$[T^*(f)](\boldsymbol{v}) := f(T(\boldsymbol{v})), \ \forall \boldsymbol{v} \in V.$$

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1. In other words, $T^*(f) = f \circ T$, i.e., a linear transformation from V to $\mathbb{F}$, i.e., belongs to V^*.
2. Moreover, the mapping T^* itself is a linear transformation: For $f,g \in W^*$,

and $\forall \boldsymbol{v} \in V$,

$$\begin{aligned}
[T^*(af + bg)](\boldsymbol{v}) &= (af + bg)[T(\boldsymbol{v})] \\
&= af(T(\boldsymbol{v})) + bg(T(\boldsymbol{v})) && \text{definition of } W^* \text{ as a vector space} \\
&= a[T^*(f)](\boldsymbol{v}) + b[T^*(g)](\boldsymbol{v}) \\
&= [aT^*(f) + bT^*(g)](\boldsymbol{v}) && \text{definition of } V^* \text{ as a vector space}
\end{aligned}$$

Proposition 5.5 Let $T : V \to W$ be a linear transformation.

1. If T is **injective**, then T^* is **surjective**.
2. If T is **surjetive**, then T^* is **injective**.

This statement is quite intuitive, since T^* reverses the dual of output into the dual of input:

$$T : V \to W$$

$$T^* : W^* \to V^*$$

Proof. We only give a proof of (2), i.e., suffices to show $\ker(T) = \{\mathbf{0}\}$.

Consider any $g \in W^*$ such that $T^*(g) = \mathbf{0}_{V^*}$. It follows that

$$[T^*(g)](\boldsymbol{v}) = \mathbf{0}_{V^*}(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in V. \Longleftrightarrow g(T(\boldsymbol{v})) = \mathbf{0}, \quad \forall \boldsymbol{v} \in V. \tag{5.4}$$

To show $g = \mathbf{0}_{W^*}$, it suffices to show $g(\boldsymbol{w}) = \mathbf{0}$ for $\forall \boldsymbol{w} \in W$. For all $\boldsymbol{w} \in W$, by the surjectivity of T, there exists $\boldsymbol{v}' \in V$ such that

$$\boldsymbol{w} = T(\boldsymbol{v}').$$

By substituting $\boldsymbol{w}$ with $T(\boldsymbol{v}')$ and (5.4), we imply

$$g(\boldsymbol{w}) = g(T(\boldsymbol{v}')) = \mathbf{0}.$$

The proof is complete. ■

Proposition 5.6 Let $T : V \to W$ be a linear transformation, and $\mathcal{A} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}, \mathcal{B} = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\}$ be the bases of V and W, respectively. Let $\mathcal{A}^* = \{f_1, \dots, f_n\}, \mathcal{B}^* = \{g_1, \dots, g_m\}$ be bases of dual spaces V^* and W^*, respectively. Then $T^* : W^* \to V^*$ admits a matrix representation

$$(T^*)_{\mathcal{A}^*\mathcal{B}^*} = \text{transpose}\left((T)_{\mathcal{B}\mathcal{A}}\right)$$

where $(T^*)_{\mathcal{A}^*\mathcal{B}^*} \in \mathbb{F}^{n \times m}$ and $(T)_{\mathcal{B}\mathcal{A}} \in \mathbb{F}^{m \times n}$

Proof. Let $(T)_{\mathcal{B}\mathcal{A}} = (\alpha_{ij})$ and $(T^*)_{\mathcal{A}^*\mathcal{B}^*} = (\beta_{ij})$. By definition of matrix representation,

$$T(\boldsymbol{v}_j) = \sum_{i=1}^{m} \alpha_{ij} \boldsymbol{w}_i, \qquad T^*(g_i) = \sum_{k=1}^{n} \beta_{ki} f_k \in V^*$$

As a result,

$$\begin{aligned}
[T^*(g_i)](\boldsymbol{v}_j) &= g_i(T(\boldsymbol{v}_j)) \\
&= g_i\left(\sum_{\ell=1}^{m} \alpha_{\ell j} \boldsymbol{w}_\ell\right) \\
&= \sum_{\ell=1}^{m} \alpha_{\ell j} g_i(\boldsymbol{w}_\ell) \\
&= \alpha_{ij}
\end{aligned}$$

and

$$\begin{aligned}
[T^*(g_i)](\boldsymbol{v}_j) &= \left(\sum_{k=1}^{n} \beta_{ki} f_k\right)(\boldsymbol{v}_j) \\
&= \sum_{k=1}^{n} \beta_{ki} f_k(\boldsymbol{v}_j) \\
&= \beta_{ji}
\end{aligned}$$

Therefore, $\beta_{ji} = \alpha_{ij}$. The proof is complete. ■

5.4.2. Relationship between Annihilator and dual of quotient spaces

■ **Example 5.5** Consider the canonical projection mapping $\pi_W : V \to V/W$ with its **adjoint** mapping:

$$(\pi_W)^* : (V/W)^* \to V^*$$

The understanding of $(\pi_W)^*$ is as follows:

1. Take $h \in (V/W)^*$ and study $(\pi_W)^*(h) \in V^*$
2. Take $\boldsymbol{v} \in V$ and understand

$$[(\pi_W)^*(h)](\boldsymbol{v}) = h(\pi_W(\boldsymbol{v})) = h(\boldsymbol{v} + W)$$

(a) In particular, for all $\boldsymbol{w} \in W \leq V$, we have

$$[(\pi_W)^*(h)](\boldsymbol{w}) = h(\boldsymbol{w} + W) = h(\mathbf{0}_{V/W}) = \mathbf{0}_{\mathbb{F}}$$

Therefore,

$$(\pi_W)^*(h) \in \mathrm{Ann}(W).$$

i.e., $(\pi_W)^*$ is a mapping from $(V/W)^*$ to $\mathrm{Ann}(W)$.

(b) By proposition (5.5), π_W is surjective implies $(\pi_W)^*$ is injective.

Combining (a) and (b), it's clear that (i.e., left as homework problem)

$$\Phi \circ \pi_W^* = \mathrm{id}_{(V/W)^*} \text{ and } \pi_W^* \circ \Phi = \mathrm{id}_{\mathrm{Ann}(W)}$$

This relationship implies Φ is an isomorphism. ■