Chapter 5

Week5

5.1. Monday for MAT3040

Reviewing.

- Dual space: the set of linear transformations from V to $\mathbb{F}$, denoted as $\text{Hom}(V,\mathbb{F})$.

- Suppose $B = \{\boldsymbol{v}_i \mid i \in I\}$ is the basis of V, define $B^* = \{f_i \mid i \in I\}$ by

$$f_i(\boldsymbol{v}_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

  Actually, the above recipe uniquely defines a linear transformation $f_i : V \to \mathbb{F}$: For any $\boldsymbol{v} \in V$, it can be written as $\boldsymbol{v} = \sum_{i \in I} \alpha_i \boldsymbol{v}_i$, and therefore

$$f_i(\boldsymbol{v}) = f_i(\sum_{i \in I} \alpha_i \boldsymbol{v}_i) = \sum_{i \in I} \alpha_i f_i(\boldsymbol{v}_i).$$

Example 5.1 Consider $V = \mathbb{R}^n, B = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$. Then we imply $B^* = \{\phi_i\}_{i=1}^n$, where ϕ_i is the mapping $V \to \mathbb{R}$ defined by

$$\phi_i \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \phi(x_1 \boldsymbol{e}_1 + \cdots + x_n \boldsymbol{e}_n) = \sum_{j=1}^n x_j \phi_i(\boldsymbol{e}_j) = x_i$$

5.1.1. Remarks on Dual Space

Proposition 5.1
1. B^* is always lienarly independent, i.e., any finite subset of B^* is linearly independent.
2. If V has finite dimension, then B^* is a basis of V^*.

Proof.
1. Suppose that

$$\alpha_1 f_{i_1} + \alpha_2 f_{i_2} + \cdots + \alpha_k f_{i_k} = \mathbf{0}_{V^*}.$$

In particular, let the input of these linear transformations be $\boldsymbol{v}_{i_1}$, we imply

$$\begin{aligned}\alpha_1 f_{i_1}(\boldsymbol{v}_{i_1}) + \alpha_2 f_{i_2}(\boldsymbol{v}_{i_1}) + \cdots + \alpha_k f_{i_k}(\boldsymbol{v}_{i_1}) &= \mathbf{0}(\boldsymbol{v}_{i_1}) \equiv \mathbf{0} \\ &= \alpha_1 \cdot 1 + \cdots + 0 \\ &= \alpha_1\end{aligned}$$

Applying the same trick, one can show that $\alpha_2 = \cdots = \alpha_k = 0$. Therefore, $\{f_{i_1}, \ldots, f_{i_k}\}$ is linearly independent.

2. Suppose that $B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ and $B^* = \{f_1, \ldots, f_n\}$. For any $f \in V^*$, construct the linear transformation

$$g := \sum_{i=1}^{n} f(\boldsymbol{v}_i) \cdot f_i \in \text{span}\{B^*\}.$$

It follows that for $j = 1, 2, \ldots, n$,

$$g(\boldsymbol{v}_j) = \sum_{i=1}^{n} f(\boldsymbol{v}_i) \cdot f_i(\boldsymbol{v}_j) = f(\boldsymbol{v}_j).$$

It's clear that $g(\boldsymbol{v}) = f(\boldsymbol{v})$ for all $\boldsymbol{v} \in V$, i.e., $f \equiv g \in \text{span}(B^*)$. Therefore B^* spans V^*, i.e., forms a basis of V^*.

■

Corollary 5.1 If $\dim(V) = n$, then $\dim(V^*) = n$.

Proof. It's eay to show the mapping defined as

$$\begin{aligned} & V \to V^* \\ \text{with} \quad & \boldsymbol{v}_i \mapsto f_i \end{aligned}$$

is an isomorphism from $V \to V^*$. Note that this constructed isomorphism depends on **the choice of basis** B in V. (We say this is not a **natural isomorphism**.) ■

R The part 2 for proposition (5.1) does not hold for V with infinite dimension. The reason is that the spanning set is defined with **finite** linear combinations. Check the example below for a counter-example.

■ **Example 5.2** Suppose that $V = \mathbb{F}[x]$, and $B^* = \{1, x, x^2, \dots,\}$ forms a basis of V. We imply that $B^* = \{\phi_0, \phi_1, \phi_2, \dots,\}$, where ϕ_i is the mapping defined as

$$\phi_i(x^j) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$$

Consider a special element $\phi \in V^*$ with $f(p(x)) = p(1)$:

$$\phi(1) = 1, \quad \phi(x) = 1, \quad \phi(x^2) = 1, \quad \cdots \quad \phi(x^n) = 1, \quad \forall n \in \mathbb{N}.$$

If following the proof in proposition (5.1), we expect that

$$g := \sum_{n=0}^{\infty} \phi(x^n)\phi_n = \sum_{n=0}^{\infty} \phi_n \in \operatorname{span}\{B^*\},$$

which is a contradiction, since $\operatorname{span}\{B^*\}$ consists of finite sum of ϕ_i's only. ■

R Therefore, if V is not finite-dimensional, we can say the cardinality of V is strictly less than the cardinality of V^*.

Any subspace of a given vector space has some gap. Now we want to describe this gap formally from the perspective of the dual space.

5.1.2. Annihilators

Definition 5.1 Let V be a vector space, $S \subseteq V$ be a subset. The **annihilator** of S is defined as

$$\text{Ann}(S) = \{f \in V^* \mid f(s) = 0, \forall s \in S\}$$

■ **Example 5.3** Consider $V = \mathbb{R}^4$, $B = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_4\}$. Let $B^* = \{f_1, \dots, f_4\}$, $S = \{\boldsymbol{e}_3, \boldsymbol{e}_4\}$.

- Then $f_1 \in \text{Ann}(S)$, since

$$f_1(\boldsymbol{e}_3) = 0, \quad f_1(\boldsymbol{e}_4) = 0$$

Indeed, any $a \cdot f_1 + b \cdot f_2 \in V^*$ is in Ann(S).

Proposition 5.2 1. The set Ann(S) is a vector subspace of V^*

2. The mapping Ann$(\cdot)$ is **inclusion-reversing**, i.e., if $W_1 \subseteq W_2 \subseteq V$, then

$$\text{Ann}(W_1) \supseteq \text{Ann}(W_2)$$

3. The mapping Ann$(\cdot)$ is **idempotent**, i.e., $\text{Ann}(S) = \text{Ann}(\text{span}(S))$.
4. If V has finite dimension, and $W \leq V$, then Ann(W) fills in the gap, i.e.,

$$\dim(W) + \dim(\text{Ann}(W)) = \dim(V)$$

Proof. 1. Suppose that $f, g \in \text{Ann}(S)$, i.e., $f(s) = g(s) = 0, \forall s \in S$. It's clear that $(af + bg) \in \text{Ann}(S)$.

2. Suppose that $f \in \text{Ann}(W_2)$, we imply $f(\boldsymbol{w}) = 0$ for any $\boldsymbol{w} \in W_2$. Therefore, $f(\boldsymbol{w}_1) = 0$ for any $\boldsymbol{w}_1 \in W_1 \subseteq W_2$, i.e., $f \in \text{Ann}(W_1)$.
3. Note that $S \subseteq \text{span}(S)$. Therefore we imply $\text{Ann}(S) \supseteq \text{Ann}(\text{span}(S))$ from part (b). It suffices to show $\text{Ann}(S) \subseteq \text{Ann}(\text{span}(S))$:

For any $f \in \text{Ann}(S)$ and any $\sum_{i=1}^{n} k_i \boldsymbol{s}_i \in \text{span}(S)$, we imply

$$\begin{aligned} f\left(\sum_{i=1}^{n} k_i \boldsymbol{s}_i\right) &= \sum_{i=1}^{n} k_i f(\boldsymbol{s}_i) \\ &= \sum_{i=1}^{n} k_i \cdot 0 \\ &= 0, \end{aligned}$$

i.e., $f \in \text{Ann}(\text{span}(S))$.

4. Let $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ be a basis of W. By basis extension, we construct a basis of V:

$$B = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_n\}.$$

Let $B^* = \{f_1, \dots, f_k, f_{k+1}, \dots, f_n\}$ be a basis of V^*. We claim that $\{f_{k+1}, \dots, f_n\}$ is a basis of $\text{Ann}(W)$:

- Firstly, f_j's are the elements in $\text{Ann}(W)$ for $j = k+1, \dots, n$, since for any $\boldsymbol{w} = \sum_{i=1}^{k} \alpha_i(\boldsymbol{v}_i) \in W$, we have

$$\begin{aligned} f_j(\boldsymbol{w}) &= \sum_{i=1}^{k} \alpha_i f_j(\boldsymbol{v}_i) \\ &= \sum_{i=1}^{k} \alpha_i \cdot 0 \\ &= 0, \quad j = k+1, k+2, \dots, n \end{aligned}$$

- Secondly, the set $\{f_{k+1}, \dots, f_n\}$ is linearly independent, since the set $B^* = \{f_1, \dots, f_n\}$ is linearly independent.

- Thirdly, $\{f_{k+1}, \dots, f_n\}$ spans $\text{Ann}(W)$: for any $g \in \text{Ann}(W) \subseteq V^*$, it can be expressed as $g = \sum_{i=1}^n \beta_i f_i$. It follows that

$$\begin{aligned}
g(\boldsymbol{v}_1) &= \sum_{i=1}^n \beta_i f_i(\boldsymbol{v}_1) = 0 \implies \beta_1 = 0 \\
&\vdots \\
g(\boldsymbol{v}_k) &= \sum_{i=1}^n \beta_i f_i(\boldsymbol{v}_k) = 0 \implies \beta_k = 0
\end{aligned}$$

Substituting $\beta_1 = \cdots = \beta_k = 0$ into $g = \sum_{i=1}^n \beta_i f_i$, we imply

$$g = \beta_{k+1} f_{k+1} + \cdots + \beta_n f_n \in \operatorname{span}\{f_{k+1}, \dots, f_n\}.$$

Therefore, $\{f_{k+1}, \dots, f_n\}$ forms a basis for $\operatorname{Ann}(W)$, i.e., $\dim(\operatorname{Ann}(W)) = n - k$.

■

R Let $W \le V$, where V has finite dimension, recall that we have obtained two relations below:

$$\begin{aligned}
\dim(\operatorname{Ann}(W)) &= \dim(V) - \dim(W) \\
\dim((V/W)^*) &= \dim(V/W) = \dim(V) - \dim(W)
\end{aligned}$$

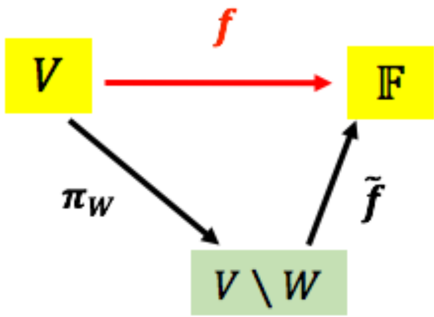
Therefore, $\dim((V/W)^*) = \dim(\operatorname{Ann}(W))$, i.e.,

$$(V/W)^* \cong \operatorname{Ann}(W).$$

The question is that can we construct an isomorphism explicitly? We claim that the mapping defined below is an isomorphism:

$$\begin{aligned}
&\operatorname{Ann}(W) \to (V/W)^* \\
\text{with} \quad &f \mapsto \tilde{f},
\end{aligned}$$

where $\tilde{f} : V/W \to \mathbb{F}$ is constructed from the **universal property I**, i.e., given the mapping $f \in \mathrm{Ann}(W)$, since $W \leq \ker(f)$, there exists $\tilde{f} : V/W \to \mathbb{F}$ such that the diagram below commutes:



i.e., $\tilde{f}(\boldsymbol{v} + W) = f(\boldsymbol{v})$.