Chapter 4

Week4

4.1. Monday for MAT3040

4.1.1. Quotient Spaces

Now we aim to divide a big vector space into many pieces of slices.

• For example, the Cartesian plane can be expressed as union of set of vertical lines as follows:

$$\mathbb{R}^{2} = \bigcup_{m \in \mathbb{R}} \left\{ \begin{pmatrix} m \\ 0 \end{pmatrix} + \operatorname{span}\{(0,1)\} \right\}$$

• Another example is that the set of integers can be expressed as union of three sets:

$$\mathbb{Z}=Z_1\cup Z_2\cup Z_3,$$

where Z_i is the set of integers *z* such that *z* mod 3 = i.

Definition 4.1 [Coset] Let V be a vector space and $W \le V$. For any element $\boldsymbol{v} \in V$, the (right) coset determined by \boldsymbol{v} is the set

$$\boldsymbol{v} + W := \{ \boldsymbol{v} + \boldsymbol{w} \mid \boldsymbol{w} \in W \}$$

For example, consider $V = \mathbb{R}^3$ and $W = \text{span}\{(1,2,0)\}$. Then the coset determined

by $\boldsymbol{v} = (5, 6, -3)$ can be written as

$$v + W = \{(5 + t, 6 + 2t, -3) \mid t \in \mathbb{R}\}$$

It's interesting that the coset determined by $\boldsymbol{v}' = \{(4,4,-3)\}$ is exactly the same as the coset shown above:

$$v' + W = \{(4 + t, 4 + 2t, -3) \mid t \in \mathbb{R}\} = v + W.$$

Therefore, write the exact expression of v + W may sometimes become tedious and hard to check the equivalence. We say v is a **representative** of a coset v + W.

Proposition 4.1 Two cosets are the same iff the subtraction for the corresponding representatives is in *W*, i.e.,

$$\boldsymbol{v}_1 + W = \boldsymbol{v}_2 + W \Longleftrightarrow \boldsymbol{v}_1 - \boldsymbol{v}_2 \in W$$

Proof. Necessity. Suppose that $v_1 + W = v_2 + W$, then $v_1 + w_1 = v_2 + w_2$ for some $w_1, w_2 \in W$, which implies

$$\boldsymbol{v}_1 - \boldsymbol{v}_2 = \boldsymbol{w}_2 - \boldsymbol{w}_1 \in W$$

Sufficiency. Suppose that $\boldsymbol{v}_1 - \boldsymbol{v}_2 = \boldsymbol{w} \in W$. It suffices to show $\boldsymbol{v}_1 + W \subseteq \boldsymbol{v}_2 + W$. For any $\boldsymbol{v}_1 + \boldsymbol{w}' \in \boldsymbol{v}_1 + W$, this element can be expressed as

$$\boldsymbol{v}_1 + \boldsymbol{w}' = (\boldsymbol{v}_2 + \boldsymbol{w}) + \boldsymbol{w}' = \boldsymbol{v}_2 + \underbrace{(\boldsymbol{w} + \boldsymbol{w}')}_{\text{belong to } W} \in \boldsymbol{v}_2 + W.$$

Therefore, $\boldsymbol{v}_1 + W \subseteq \boldsymbol{v}_2 + W$. Similarly we can show that $\boldsymbol{v}_2 + W \subseteq \boldsymbol{v}_1 + W$.

Exercise: Two cosets with representatives v_1, v_2 have no intersection iff $v_1 - v_2 \notin W$. **Definition 4.2** [Quotient Space] The **quotient space** of V by the subspace W, is the

collection of all cosets $\boldsymbol{v} + W$, denoted by V/W.

To make the quotient space a vector space structure, we define the addition and scalar

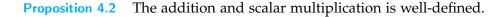
multiplication on V/W by:

$$(\boldsymbol{v}_1 + W) + (\boldsymbol{v}_2 + W) := (\boldsymbol{v}_1 + \boldsymbol{v}_2) + W$$

 $\alpha \cdot (\boldsymbol{v} + W) := (\alpha \cdot \boldsymbol{v}) + W$

For example, consider $V = \mathbb{R}^2$ and $W = \text{span}\{(0,1)\}$. Then note that

$$\left(\begin{pmatrix} 1\\0 \end{pmatrix} + W \right) + \left(\begin{pmatrix} 2\\0 \end{pmatrix} + W \right) = \left(\begin{pmatrix} 3\\0 \end{pmatrix} + W \right)$$
$$\pi \cdot \left(\begin{pmatrix} 1\\0 \end{pmatrix} + W \right) = \left(\begin{pmatrix} \pi\\0 \end{pmatrix} + W \right)$$



Proof. 1. Suppose that

$$\begin{cases} \boldsymbol{v}_1 + W = \boldsymbol{v}_1' + W \\ \boldsymbol{v}_2 + W = \boldsymbol{v}_2' + W \end{cases}, \tag{4.1}$$

and we need to show that $(\boldsymbol{v}_1 + \boldsymbol{v}_2) + W = (\boldsymbol{v}_1' + \boldsymbol{v}_2') + W$. From (4.1) and proposition (4.1), we imply

$$v_1 - v_1' \in W$$
, $v_2 - v_2' \in W$

which implies

$$(v_1 - v_1') + (v_2 - v_2') = (v_1 + v_2) - (v_1' + v_2') \in W$$

By proposition (4.1) again we imply $(\boldsymbol{v}_1 + \boldsymbol{v}_2) + W = (\boldsymbol{v}_1' + \boldsymbol{v}_2') + W$

2. For scalar multiplication, similarly, we can show that $\boldsymbol{v}_1 + W = \boldsymbol{v}'_1 + W$ implies $\alpha \boldsymbol{v}_1 + W = \alpha \boldsymbol{v}'_1 + W$ for all $\alpha \in \mathbb{F}$.

Proposition 4.3 The canonical projection mapping

$$\pi_W: V o V/W,$$
 $oldsymbol{v} \mapsto oldsymbol{v} + W,$

is a surjective linear transformation with ker(π_W) = *W*.

Proof. 1. First we show that $ker(\pi_W) = W$:

$$\pi_W(\boldsymbol{v}) = 0 \implies \boldsymbol{v} + W = \boldsymbol{0}_{V/W} \implies \boldsymbol{v} + W = \boldsymbol{0} + W \implies \boldsymbol{v} = (\boldsymbol{v} - \boldsymbol{0}) \in W$$

Here note that the zero element in the quotient space V/W is the coset with representative **0**.

- 2. For any $\boldsymbol{v}_0 + W \in V/W$, we can construct $\boldsymbol{v}_0 \in V$ such that $\pi_W(\boldsymbol{v}_0) = \boldsymbol{v}_0 + W$. Therefore the mapping π_W is surjective.
- 3. To show the mapping π_W is a linear transformation, note that

$$\pi_{W}(\alpha \boldsymbol{v}_{1} + \beta \boldsymbol{v}_{2}) = (\alpha \boldsymbol{v}_{1} + \beta \boldsymbol{v}_{2}) + W$$
$$= (\alpha \boldsymbol{v}_{1} + W) + (\beta \boldsymbol{v}_{2} + W)$$
$$= \alpha (\boldsymbol{v}_{1} + W) + \beta (\boldsymbol{v}_{2} + W)$$
$$= \alpha \pi_{W}(\boldsymbol{v}_{1}) + \beta \pi_{W}(\boldsymbol{v}_{2})$$

4.1.2. First Isomorphism Theorem

The key of linear algebra is to solve the linear system Ax = b with $A \in \mathbb{R}^{m \times n}$. The general step for solving this linear system is as follows:

- 1. Find the solution set for Ax = 0, i.e., the set ker(A)
- 2. Find a particular solution \mathbf{x}_0 such that $A\mathbf{x}_0 = \mathbf{b}$.

Then the general solution set to this linear system is $\boldsymbol{x}_0 + \ker(\boldsymbol{A})$, which is a coset in

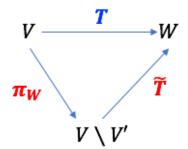
the space $\mathbb{R}^n / \ker(\mathbf{A})$. Therefore, to solve the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ suffices to study the quotient space $\mathbb{R}^n / \ker(\mathbf{A})$:

Proposition 4.4 — Universal Property I. Suppose that $T: V \to W$ is a linear transformation, and that $V' \leq \ker(T)$. Then the mapping

$$ilde{T}: V/V' \to W$$

 $oldsymbol{v} + V' \mapsto T(oldsymbol{v})$

is a well-defined linear transformation. As a result, the diagram below commutes:



In other words, we have $T = \tilde{T} \circ \pi_W$.

Proof. First we show the well-definedness. Suppose that $\boldsymbol{v}_1 + V' = \boldsymbol{v}_2 + V'$ and suffices to show $\tilde{T}(\boldsymbol{v}_1 + V') = \tilde{T}(\boldsymbol{v}_2 + V')$, i.e., $T(\boldsymbol{v}_1) = T(\boldsymbol{v}_2)$. By proposition (4.1), we imply

$$\boldsymbol{v}_1 - \boldsymbol{v}_2 \in V' \leq \ker(T) \implies T(\boldsymbol{v}_1 - \boldsymbol{v}_2) = \mathbf{0} \implies T(\boldsymbol{v}_1) - T(\boldsymbol{v}_2) = \mathbf{0}.$$

Then we show (T) is a linear transformation:

$$\begin{split} \tilde{T}(\alpha(\boldsymbol{v}_1 + V') + \beta(\boldsymbol{v}_2 + V')) &= \tilde{T}((\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2) + V') \\ &= T(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2) \\ &= \alpha T(\boldsymbol{v}_1) + \beta T(\boldsymbol{v}_2) \\ &= \alpha \tilde{T}(\boldsymbol{v}_1 + V') + \beta \tilde{T}(\boldsymbol{v}_2 + V') \end{split}$$

Actually, if we let $V' = \ker(T)$, the mapping $\tilde{T} : V/V' \to T(V)$ forms an isomorphism, In particular, if further *T* is surjective, then T(V) = W, i.e., the mapping $\tilde{T} : V/V' \to W$ forms an isomorphism.

Theorem 4.1 — First Isomorphism Theorem. Let $T: V \to W$ be a surjective linear transformation. Then the mapping

$$\tilde{T}: V / \ker(T) \to W$$

 $\boldsymbol{v} + \ker(T) \mapsto T(\boldsymbol{v})$

is an isomorphism.

Proof. Injectivity. Suppose that $\tilde{T}(\boldsymbol{v}_1 + \ker(T)) = \tilde{T}(\boldsymbol{v}_2 + \ker(T))$, then we imply

$$T(\boldsymbol{v}_1) = T(\boldsymbol{v}_2) \implies T(\boldsymbol{v}_1 - \boldsymbol{v}_2) = \boldsymbol{0}_W \implies \boldsymbol{v}_1 - \boldsymbol{v}_2 \in \ker(T),$$

i.e., $v_1 + \ker(T) = v_2 + \ker(T)$.

Surjectivity. For $\boldsymbol{w} \in W$, due to the surjectivity of T, we can find a \boldsymbol{v}_0 such that $T(\boldsymbol{v}_0) = \boldsymbol{w}$. Therefore, we can construct a set $\boldsymbol{v}_0 + \ker(T)$ such that

$$\tilde{T}(\boldsymbol{v}_0 + \ker(T)) = \boldsymbol{w}.$$