Chapter 4

Week4

4.1. Monday for MAT3040

4.1.1. Quotient Spaces

Now we aim to divide a big **vector space** into many pieces of slices.

- For example, the Cartesian plane can be expressed as union of set of vertical lines as follows:
$$\mathbb{R}^2 = \bigcup_{m\in\mathbb{R}} \left\{ \begin{pmatrix} m \\ 0 \end{pmatrix} + \operatorname{span}\{(0,1)\} \right\}$$

- Another example is that the set of integers can be expressed as union of three sets:
$$\mathbb{Z} = Z_1 \cup Z_2 \cup Z_3,$$
where Z_i is the set of integers z such that $z \bmod 3 = i$.

Definition 4.1 [Coset] Let V be a vector space and $W \le V$. For any element $\boldsymbol{v} \in V$, the **(right) coset** determined by $\boldsymbol{v}$ is the set

$$\boldsymbol{v} + W := \{\boldsymbol{v} + \boldsymbol{w} \mid \boldsymbol{w} \in W\}$$

For example, consider $V = \mathbb{R}^3$ and $W = \operatorname{span}\{(1,2,0)\}$. Then the coset determined by $\boldsymbol{v} = (5,6,-3)$ can be written as

$$\boldsymbol{v} + W = \{(5+t, 6+2t, -3) \mid t \in \mathbb{R}\}$$

It's interesting that the coset determined by $\boldsymbol{v}' = \{(4,4,-3)\}$ is exactly the same as the coset shown above:

$$\boldsymbol{v}' + W = \{(4+t, 4+2t, -3) \mid t \in \mathbb{R}\} = \boldsymbol{v} + W.$$

Therefore, write the exact expression of $\boldsymbol{v} + W$ may sometimes become tedious and hard to check the equivalence. We say $\boldsymbol{v}$ is a **representative** of a coset $\boldsymbol{v} + W$.

Proposition 4.1 Two cosets are the same iff the subtraction for the corresponding representatives is in W, i.e.,

$$\boldsymbol{v}_1 + W = \boldsymbol{v}_2 + W \iff \boldsymbol{v}_1 - \boldsymbol{v}_2 \in W$$

Proof. Necessity. Suppose that $\boldsymbol{v}_1 + W = \boldsymbol{v}_2 + W$, then $\boldsymbol{v}_1 + \boldsymbol{w}_1 = \boldsymbol{v}_2 + \boldsymbol{w}_2$ for some $\boldsymbol{w}_1, \boldsymbol{w}_2 \in W$, which implies

$$\boldsymbol{v}_1 - \boldsymbol{v}_2 = \boldsymbol{w}_2 - \boldsymbol{w}_1 \in W$$

Sufficiency. Suppose that $\boldsymbol{v}_1 - \boldsymbol{v}_2 = \boldsymbol{w} \in W$. It suffices to show $\boldsymbol{v}_1 + W \subseteq \boldsymbol{v}_2 + W$. For any $\boldsymbol{v}_1 + \boldsymbol{w}' \in \boldsymbol{v}_1 + W$, this element can be expressed as

$$\boldsymbol{v}_1 + \boldsymbol{w}' = (\boldsymbol{v}_2 + \boldsymbol{w}) + \boldsymbol{w}' = \boldsymbol{v}_2 + \underbrace{(\boldsymbol{w} + \boldsymbol{w}')}_{\text{belong to } W} \in \boldsymbol{v}_2 + W.$$

Therefore, $\boldsymbol{v}_1 + W \subseteq \boldsymbol{v}_2 + W$. Similarly we can show that $\boldsymbol{v}_2 + W \subseteq \boldsymbol{v}_1 + W$. ■

Exercise: Two cosets with representatives $\boldsymbol{v}_1, \boldsymbol{v}_2$ have no intersection iff $\boldsymbol{v}_1 - \boldsymbol{v}_2 \notin W$.

Definition 4.2 [Quotient Space] The **quotient space** of V by the subspace W, is the collection of all cosets $\boldsymbol{v} + W$, denoted by V/W.

To make the quotient space a vector space structure, we define the addition and scalar multiplication on V/W by:

$$(\boldsymbol{v}_1 + W) + (\boldsymbol{v}_2 + W) := (\boldsymbol{v}_1 + \boldsymbol{v}_2) + W$$
$$\alpha \cdot (\boldsymbol{v} + W) := (\alpha \cdot \boldsymbol{v}) + W$$

For example, consider $V = \mathbb{R}^2$ and $W = \operatorname{span}\{(0,1)\}$. Then note that

$$\left(\begin{pmatrix}1\\0\end{pmatrix} + W\right) + \left(\begin{pmatrix}2\\0\end{pmatrix} + W\right) = \left(\begin{pmatrix}3\\0\end{pmatrix} + W\right)$$
$$\pi \cdot \left(\begin{pmatrix}1\\0\end{pmatrix} + W\right) = \left(\begin{pmatrix}\pi\\0\end{pmatrix} + W\right)$$

Proposition 4.2 The addition and scalar multiplication is well-defined.

Proof. 1. Suppose that

$$\begin{cases} \boldsymbol{v}_1 + W = \boldsymbol{v}_1' + W \\ \boldsymbol{v}_2 + W = \boldsymbol{v}_2' + W \end{cases}, \tag{4.1}$$

and we need to show that $(\boldsymbol{v}_1 + \boldsymbol{v}_2) + W = (\boldsymbol{v}_1' + \boldsymbol{v}_2') + W$.

From (4.1) and proposition (4.1), we imply

$$\boldsymbol{v}_1 - \boldsymbol{v}_1' \in W, \quad \boldsymbol{v}_2 - \boldsymbol{v}_2' \in W$$

which implies

$$(\boldsymbol{v}_1 - \boldsymbol{v}_1') + (\boldsymbol{v}_2 - \boldsymbol{v}_2') = (\boldsymbol{v}_1 + \boldsymbol{v}_2) - (\boldsymbol{v}_1' + \boldsymbol{v}_2') \in W$$

By proposition (4.1) again we imply $(\boldsymbol{v}_1 + \boldsymbol{v}_2) + W = (\boldsymbol{v}_1' + \boldsymbol{v}_2') + W$

2. For scalar multiplication, similarly, we can show that $\boldsymbol{v}_1 + W = \boldsymbol{v}_1' + W$ implies $\alpha\boldsymbol{v}_1 + W = \alpha\boldsymbol{v}_1' + W$ for all $\alpha \in \mathbb{F}$.

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Proposition 4.3 The canonical projection mapping

$$\begin{aligned}\pi_W : & V \to V/W, \\ & \boldsymbol{v} \mapsto \boldsymbol{v} + W,\end{aligned}$$

is a **surjective linear transformation** with $\ker(\pi_W) = W$.

Proof. 1. First we show that $\ker(\pi_W) = W$:

$$\pi_W(\boldsymbol{v}) = 0 \implies \boldsymbol{v} + W = \mathbf{0}_{V/W} \implies \boldsymbol{v} + W = \mathbf{0} + W \implies \boldsymbol{v} = (\boldsymbol{v} - \mathbf{0}) \in W$$

Here note that the zero element in the quotient space V/W is the coset with representative $\mathbf{0}$.

2. For any $\boldsymbol{v}_0 + W \in V/W$, we can construct $\boldsymbol{v}_0 \in V$ such that $\pi_W(\boldsymbol{v}_0) = \boldsymbol{v}_0 + W$. Therefore the mapping π_W is surjective.
3. To show the mapping π_W is a linear transformation, note that

$$\begin{aligned}\pi_W(\alpha\boldsymbol{v}_1 + \beta\boldsymbol{v}_2) &= (\alpha\boldsymbol{v}_1 + \beta\boldsymbol{v}_2) + W \\ &= (\alpha\boldsymbol{v}_1 + W) + (\beta\boldsymbol{v}_2 + W) \\ &= \alpha(\boldsymbol{v}_1 + W) + \beta(\boldsymbol{v}_2 + W) \\ &= \alpha\pi_W(\boldsymbol{v}_1) + \beta\pi_W(\boldsymbol{v}_2)\end{aligned}$$

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4.1.2. First Isomorphism Theorem

The key of linear algebra is to solve the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ with $\boldsymbol{A} \in \mathbb{R}^{m \times n}$. The general step for solving this linear system is as follows:

1. Find the solution set for $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$, i.e., the set $\ker(\boldsymbol{A})$
2. Find a particular solution $\boldsymbol{x}_0$ such that $\boldsymbol{A}\boldsymbol{x}_0 = \boldsymbol{b}$.

Then the general solution set to this linear system is $\boldsymbol{x}_0 + \ker(\boldsymbol{A})$, which is a coset in the space $\mathbb{R}^n / \ker(\boldsymbol{A})$. Therefore, to solve the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ suffices to study the quotient space $\mathbb{R}^n / \ker(\boldsymbol{A})$:

Proposition 4.4 — Universal Property I. Suppose that $T : V \to W$ is a linear transformation, and that $V' \le \ker(T)$. Then the mapping

$$\begin{aligned} \tilde{T} : V/V' &\to W \\ \boldsymbol{v} + V' &\mapsto T(\boldsymbol{v}) \end{aligned}$$

is a well-defined linear transformation. As a result, the diagram below commutes:

$$\begin{tikzcd} V \arrow[rr, "T"] \arrow[dr, "\pi_W"'] & & W \\ & V \setminus V' \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

In other words, we have $T = \tilde{T} \circ \pi_W$.

Proof. First we show the well-definedness. Suppose that $\boldsymbol{v}_1 + V' = \boldsymbol{v}_2 + V'$ and suffices to show $\tilde{T}(\boldsymbol{v}_1 + V') = \tilde{T}(\boldsymbol{v}_2 + V')$, i.e., $T(\boldsymbol{v}_1) = T(\boldsymbol{v}_2)$. By proposition (4.1), we imply

$$\boldsymbol{v}_1 - \boldsymbol{v}_2 \in V' \le \ker(T) \implies T(\boldsymbol{v}_1 - \boldsymbol{v}_2) = \boldsymbol{0} \implies T(\boldsymbol{v}_1) - T(\boldsymbol{v}_2) = \boldsymbol{0}.$$

Then we show $\tilde{(T)}$ is a linear transformation:

$$\begin{aligned} \tilde{T}(\alpha(\boldsymbol{v}_1 + V') + \beta(\boldsymbol{v}_2 + V')) &= \tilde{T}((\alpha\boldsymbol{v}_1 + \beta\boldsymbol{v}_2) + V') \\ &= T(\alpha\boldsymbol{v}_1 + \beta\boldsymbol{v}_2) \\ &= \alpha T(\boldsymbol{v}_1) + \beta T(\boldsymbol{v}_2) \\ &= \alpha\tilde{T}(\boldsymbol{v}_1 + V') + \beta\tilde{T}(\boldsymbol{v}_2 + V') \end{aligned}$$

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Actually, if we let $V' = \ker(T)$, the mapping $\tilde{T} : V/V' \to T(V)$ forms an isomorphism, In particular, if further T is surjective, then $T(V) = W$, i.e., the mapping $\tilde{T} : V/V' \to W$ forms an isomorphism.

Theorem 4.1 — First Isomorphism Theorem. Let $T : V \to W$ be a surjective linear transformation. Then the mapping

$$\begin{aligned} \tilde{T} : V/\ker(T) &\to W \\ \boldsymbol{v} + \ker(T) &\mapsto T(\boldsymbol{v}) \end{aligned}$$

is an isomorphism.

Proof. Injectivity. Suppose that $\tilde{T}(\boldsymbol{v}_1 + \ker(T)) = \tilde{T}(\boldsymbol{v}_2 + \ker(T))$, then we imply

$$T(\boldsymbol{v}_1) = T(\boldsymbol{v}_2) \implies T(\boldsymbol{v}_1 - \boldsymbol{v}_2) = \mathbf{0}_W \implies \boldsymbol{v}_1 - \boldsymbol{v}_2 \in \ker(T),$$

i.e., $\boldsymbol{v}_1 + \ker(T) = \boldsymbol{v}_2 + \ker(T)$.

Surjectivity. For $\boldsymbol{w} \in W$, due to the surjectivity of T, we can find a $\boldsymbol{v}_0$ such that $T(\boldsymbol{v}_0) = \boldsymbol{w}$. Therefore, we can construct a set $\boldsymbol{v}_0 + \ker(T)$ such that

$$\tilde{T}(\boldsymbol{v}_0 + \ker(T)) = \boldsymbol{w}.$$

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