3.4. Wednesday for MAT3040

3.4.1. Remarks for the Change of Basis

Reviewing.

- $[\cdot]_{\mathcal{A}} : V \to \mathbb{F}^n$ denotes coordinate vector mapping
- Change of Basis matrix: $\mathcal{C}_{\mathcal{A}',\mathcal{A}}$
- $T : V \to W$, $\mathcal{A} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ and $\boldsymbol{B} = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\}$.

  $\mathrm{Hom}_{\mathbb{F}}(V, W) \to M_{m\times n}(\mathbb{F})$

■ **Example 3.10** Let $V = \mathbb{P}_3[x]$ and $\mathcal{A} = \{1, x, x^2, x^3\}$.

Let $T : V \to V$ defined as $p(x) \mapsto p'(x)$:

$$\begin{cases} T(1) = 0\cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3 \\ T(x) = 1\cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3 \\ T(x^2) = 0\cdot 1 + 2\cdot x + 0\cdot x^2 + 0\cdot x^3 \\ T(x^3) = 0\cdot 1 + 0\cdot x + 3\cdot x^2 + 0\cdot x^3 \end{cases}$$

We can define the change of basis matrix for a linear transformation T as well, w.r.t. $\mathcal{A}$ and $\mathcal{A}$:

$$\mathcal{C}_{\mathcal{A},\mathcal{A}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Also, we can define a different basis $\mathcal{A}' = \{x^3, x^2, x, 1\}$ for the output space for T, say $T : V_{\mathcal{A}} \to V_{\mathcal{A}'}$:

$$(T)_{\mathcal{A},\mathcal{A}'} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Our observation is that the corresponding coordinate vectors before and after linear transformation admits a matrix multiplication:

$$(2x^2+4x^3) \xrightarrow{T} ((4x+12x^2))$$

$$(2x^2+4x^3)_{\mathcal{A}} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 4 \end{pmatrix} \qquad (4x+12x^2)_{\mathcal{A}} = \begin{pmatrix} 0 \\ 4 \\ 12 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 12 \\ 0 \end{pmatrix}$$

$$\mathcal{C}_{\mathcal{A}\mathcal{A}} \cdot (2x^2+4x^3)_{\mathcal{A}} = (4x+12x^2)_{\mathcal{A}}$$

■

Theorem 3.3 — **Matrix Representation.** Let $T : V \to W$ be a linear transformation of finite dimensional vector sapces. Let $\mathcal{A}, \mathcal{B}$ the ordered basis of V, W, respectively. Then the following diagram holds:

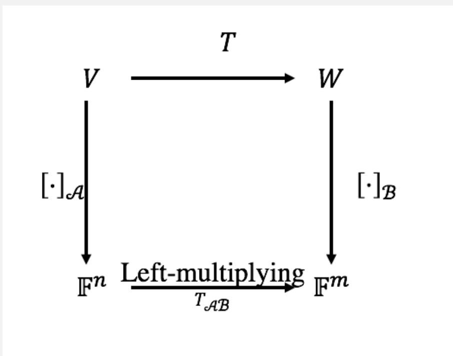

Figure 3.2: Diagram for the matrix reprentation, where $n := \dim(V)$ and $m := \dim(W)$

namely, for any $\boldsymbol{v} \in V$,

$$(T)_{\mathcal{B},\mathcal{A}}(\boldsymbol{v})_{\mathcal{A}} = (T\boldsymbol{v})_{\mathcal{B}}$$

Therefore, we can compute $T\boldsymbol{v}$ by matrix multiplication.

R Linear transformation corresponds to coordinate matrix multiplication.

Proof. Suppose $\mathcal{A} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ and $\mathcal{B} = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$. The proof of this theorem follows the same procedure of that in Theorem (3.1)

1. We show this result for $\boldsymbol{v} = \boldsymbol{v}_j$ first:

$$\text{LHS} = [\alpha_{ij}]\boldsymbol{e}_j = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix}$$

$$\text{RHS} = (T\boldsymbol{v}_j)_{\mathcal{B}} = \left(\sum_{i=1}^{m} \alpha_{ij}\boldsymbol{w}_i\right)_{\mathcal{B}} = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix}$$

2. Then we show the theorem holds for any $\boldsymbol{v} := \sum_{j=1}^{n} r_j\boldsymbol{v}_j$ in V:

$$
\begin{aligned}
(T)_{\mathcal{B}\mathcal{A}}(\boldsymbol{v})_{\mathcal{A}} &= (T)_{\mathcal{B}\mathcal{A}}\left(\sum_{j=1}^{n} r_j\boldsymbol{v}_j\right)_{\mathcal{A}} && \text{(3.8a)} \\
&= (T)_{\mathcal{B}\mathcal{A}}\left(\sum_{j=1}^{n} r_j(\boldsymbol{v}_j)_{\mathcal{A}}\right) && \text{(3.8b)} \\
&= \sum_{j=1}^{n} r_j(T)_{\mathcal{B}\mathcal{A}}(\boldsymbol{v}_j)_{\mathcal{A}} && \text{(3.8c)} \\
&= \sum_{j=1}^{n} r_j(T\boldsymbol{v}_j)_{\mathcal{B}} && \text{(3.8d)} \\
&= \left(\sum_{j=1}^{n} r_j(T\boldsymbol{v}_j)\right)_{\mathcal{B}} && \text{(3.8e)} \\
&= \left[T(\sum_{j=1}^{n} r_j\boldsymbol{v}_j)\right]_{\mathcal{B}} && \text{(3.8f)} \\
&= (T\boldsymbol{v})_{\mathcal{B}} && \text{(3.8g)}
\end{aligned}
$$

The justification for (3.8a) is similar to that shown in Theorem (3.1). The proof is complete.

■

R Consider a special case for Theorem (3.3), i.e., $T = \text{id}$ and $\mathcal{A}, \mathcal{A}'$ are two ordered basis for the input and output space, respectively. Then the result in Theorem (3.3) implies

$$C_{\mathcal{A}',\mathcal{A}}(\boldsymbol{v})_{\mathcal{A}} = (\boldsymbol{v})_{\mathcal{A}'}$$

i.e., the matrix representation theorem (3.3) is a general case for the change of basis theorem (3.1)

Proposition 3.6 — Functionality. Suppose V, W, U are finite dimensional vector spaces, and let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be the ordered basis for V, W, U, respectively. Suppose that

$$T : V \to W, \quad S : W \to U$$

are given two linear transformations, then

$$(S \circ T)_{\mathcal{C},\mathcal{A}} = (S)_{\mathcal{C},\mathcal{B}}(T)_{\mathcal{B},\mathcal{A}}$$

Composition of linear transformation corresponds to the multiplication of change of basis matrices.

Proof. Suppose the ordered basis $\mathcal{A} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$, $\mathcal{B} = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\}$, $\mathcal{C} = \{\boldsymbol{u}_1, \dots, \boldsymbol{u}_p\}$. By defintion of change of basis matrices,

$$T(\boldsymbol{v}_j) = \sum_i (T_{\mathcal{B},\mathcal{A}})_{ij} \boldsymbol{w}_i$$

$$S(\boldsymbol{w}_i) = \sum_k (S_{\mathcal{C},\mathcal{B}})_{ki} \boldsymbol{u}_k$$

We start from the j-th column of $(S \circ T)_{\mathcal{C},\mathcal{A}}$ for $j = 1, \dots, n$, namely

$$
\begin{aligned}
(S \circ T)_{\mathcal{C},\mathcal{A}}(\boldsymbol{v}_j)_{\mathcal{A}} &= (S \circ T(\boldsymbol{v}_j))_{\mathcal{C}} && \text{(3.9a)} \\
&= \left[S \circ \left(\sum_i (T_{\mathcal{B},\mathcal{A}})_{ij} \boldsymbol{w}_i \right) \right]_{\mathcal{C}} && \text{(3.9b)} \\
&= \sum_i (T_{\mathcal{B},\mathcal{A}})_{ij} \left(S(\boldsymbol{w}_i) \right)_{\mathcal{C}} && \text{(3.9c)} \\
&= \sum_i (T_{\mathcal{B},\mathcal{A}})_{ij} \left(\sum_k (S_{\mathcal{C},\mathcal{B}})_{ki} \boldsymbol{u}_k \right)_{\mathcal{C}} && \text{(3.9d)} \\
&= \sum_k \sum_i (S_{\mathcal{C},\mathcal{B}})_{ki} (T_{\mathcal{B},\mathcal{A}})_{ij} (\boldsymbol{u}_k)_{\mathcal{C}} && \text{(3.9e)} \\
&= \sum_k (S_{\mathcal{C},\mathcal{B}} T_{\mathcal{B},\mathcal{A}})_{kj} (\boldsymbol{u}_k)_{\mathcal{C}} && \text{(3.9f)} \\
&= \sum_k (S_{\mathcal{C},\mathcal{B}} T_{\mathcal{B},\mathcal{A}})_{kj} \boldsymbol{e}_k && \text{(3.9g)} \\
&= j\text{-th column of } [S_{\mathcal{C}\mathcal{B}} T_{\mathcal{B},\mathcal{A}}] && \text{(3.9h)}
\end{aligned}
$$

where (3.9a) is by the result in theorem (3.3); (3.9b) and (3.9d) follows from definitions of $T(\boldsymbol{v}_j)$ and $S(\boldsymbol{w}_i)$; (3.9c) and (3.9e) follows from the linearity of $\mathcal{C}$; (3.9f) follows from the matrix multiplication definition; (3.9g) is because $(\boldsymbol{u}_k)_{\mathcal{C}} = \boldsymbol{e}_k$.

Therefore, $(S \circ T)_{\mathcal{C}\mathcal{A}}$ and $(S_{\mathcal{C},\mathcal{B}})(T_{\mathcal{B},\mathcal{A}})$ share the same j-th column, and thus equal to each other. ■

Corollary 3.2 Suppose that S and T are two identity mappings $V \to V$, and consider $(S)_{\mathcal{A}'\mathcal{A}}$ and $(T)_{\mathcal{A},\mathcal{A}'}$ in proposition (3.6), then

$$(S \circ T)_{\mathcal{A}',\mathcal{A}'} = (S)_{\mathcal{A}'\mathcal{A}} (T)_{\mathcal{A},\mathcal{A}'}$$

Therefore,

$$\text{Identity matrix} = \mathcal{C}_{\mathcal{A}',\mathcal{A}} \mathcal{C}_{\mathcal{A},\mathcal{A}'}$$

Proposition 3.7 Let $T : V \to W$ with $\dim(V) = n, \dim(W) = m$, and let

- $\mathcal{A}, \mathcal{A}'$ be ordered basis of V
- $\mathcal{B}, \mathcal{B}'$ be ordered basis of W

then the change of basis matrices admit the relation

$$(T)_{\mathcal{B}',\mathcal{A}'} = \mathcal{C}_{\mathcal{B}',\mathcal{B}}(T)_{\mathcal{B},\mathcal{A}}\mathcal{C}_{\mathcal{A}\mathcal{A}'} \tag{3.10}$$

Here note that $(T)_{\mathcal{B}',\mathcal{A}'}, (T)_{\mathcal{B},\mathcal{A}} \in \mathbb{F}^{m\times n}$; $\mathcal{C}_{\mathcal{B}',\mathcal{B}} \in \mathbb{F}^{m\times m}$; and $\mathcal{C}_{\mathcal{A}\mathcal{A}'} \in \mathbb{F}^{n\times n}$.

Proof. Let $\mathcal{A} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}, \mathcal{A}' = \{\boldsymbol{v}'_1, \dots, \boldsymbol{v}'_n\}$. Consider simplifying the j-th column for the LHS and RHS of (3.10) and showing they are equal:

$$\begin{aligned}
\text{LHS} &= (T)_{\mathcal{B}',\mathcal{A}'}\boldsymbol{e}_j \\
&= (T)_{\mathcal{B}',\mathcal{A}'}(\boldsymbol{v}'_j)_{\mathcal{A}'} \\
&= (T\boldsymbol{v}'_j)_{\mathcal{B}'}
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \mathcal{C}_{\mathcal{B}',\mathcal{B}}(T)_{\mathcal{B},\mathcal{A}}\mathcal{C}_{\mathcal{A}\mathcal{A}'}\boldsymbol{e}_j \\
&= \mathcal{C}_{\mathcal{B}',\mathcal{B}}(T)_{\mathcal{B},\mathcal{A}}\mathcal{C}_{\mathcal{A}\mathcal{A}'}(\boldsymbol{v}'_j)_{\mathcal{A}'} \\
&= \mathcal{C}_{\mathcal{B}',\mathcal{B}}(T)_{\mathcal{B},\mathcal{A}}(\boldsymbol{v}'_j)_{\mathcal{A}} \\
&= \mathcal{C}_{\mathcal{B}',\mathcal{B}}(T\boldsymbol{v}'_j)_{\mathcal{B}} \\
&= (T\boldsymbol{v}'_j)_{\mathcal{B}'}
\end{aligned}$$

■

R Let $T : V \to V$ be a linear operator with $\mathcal{A}, \mathcal{A}'$ being two ordered basisof V, then

$$(T)_{\mathcal{A}'\mathcal{A}'} = \mathcal{C}_{\mathcal{A}',\mathcal{A}}(T)_{\mathcal{A}\mathcal{A}}\mathcal{C}_{\mathcal{A},\mathcal{A}'} = (\mathcal{C}_{\mathcal{A},\mathcal{A}'})^{-1}(T)_{\mathcal{A}\mathcal{A}}\mathcal{C}_{\mathcal{A},\mathcal{A}'}$$

Therefore, the change of basis matrices $(T)_{\mathcal{A}'\mathcal{A}'}$ and $(T)_{\mathcal{A}\mathcal{A}}$ are similar to each other, which means they share the same eigenvalues, determinant, trace.

Therefore, two similar matrices cooresponds to same linear transformation using different basis.