Chapter 3

Week3

3.1. Monday for MAT3040

Reviewing.

1. Complementation. Suppose $\dim(V) = n < \infty$, then $W \leq V$ implies that there exists W' such that
$$W \oplus W' = V.$$
2. Given the linear transformation $T : V \to W$, define the set $\ker(T)$ and $\operatorname{Im}(T)$.
3. Isomorphism of vector spaces: $T : V \cong W$
4. Rank-Nullity Theorem

3.1.1. Remarks on Isomorphism

Proposition 3.1 If $T : V \to W$ is an isomorphism, then

1. the set $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is linearly independent in V if and only if $\{T\boldsymbol{v}_1, \dots, T\boldsymbol{v}_k\}$ is linearly independent.
2. The same goes if we replace the linearly independence by spans.
3. If $\dim(V) = n$, then $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ forms a basis of V if and only if $\{T\boldsymbol{v}_1, \dots, T\boldsymbol{v}_n\}$ forms a basis of W. In particular, $\dim(V) = \dim(W)$.
4. Two vector spaces with finite dimensions are isomorphic if and only if they have the same dimension:

Proof. It suffices to show the reverse direction. Let $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ and $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ be two basis of V, W, respectively. Define the linear transformation $T: V \to W$ by

$$T(a_1\boldsymbol{v}_1 + \cdots + a_n\boldsymbol{v}_n) = a_1\boldsymbol{w}_1 + \cdots + a_n\boldsymbol{w}_n$$

Then T is surjective since $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ spans W; T is injective since $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ is linearly independent. ■

3.1.2. Change of Basis and Matrix Representation

Definition 3.1 [Coordinate Vector] Let V be a finite dimensional vector space and $B = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ an **ordered** basis of V. Any vector $\boldsymbol{v} \in V$ can be uniquely written as

$$\boldsymbol{v} = \alpha_1\boldsymbol{v}_1 + \cdots + \alpha_n\boldsymbol{v}_n,$$

Therefore we define the map $[\cdot]_{\mathcal{B}} : V \to \mathbb{F}^n$, which maps any vector in $\boldsymbol{v}$ into its **coordinate vector**:

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

R Note that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ and $\{\boldsymbol{v}_2, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ are distinct ordered basis.

■ **Example 3.1** Given $V = M_{2\times 2}(\mathbb{F})$ and the ordered basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \right\}$$

Any matrix has the coordinate vector w.r.t. $\mathcal{B}$, i.e.,

$$\left[\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 3 \end{pmatrix}$$

However, if given another ordered basis

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \right\},$$

the matrix may have the different coordinate vector w.r.t. $\mathcal{B}_1$:

$$\left[\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\right]_{\mathcal{B}_1} = \begin{pmatrix} 4 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

■

Theorem 3.1 The mapping $[\cdot]_{\mathcal{B}} : V \to \mathbb{F}^n$ is an isomorphism.

Proof. 1. First show the operator $[\cdot]_{\mathcal{B}}$ is well-defined, i.e., the same input gives the same output. Suppose that

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \quad [\boldsymbol{v}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1' \\ \vdots \\ \alpha_n' \end{pmatrix},$$

then we imply

$$\begin{aligned} \boldsymbol{v} &= \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n \\ &= \alpha_1' \boldsymbol{v}_1 + \cdots + \alpha_n' \boldsymbol{v}_n. \end{aligned}$$

By the uniqueness of coordinates, we imply $\alpha_i = \alpha_i'$ for $i = 1, \dots, n$.

2. It's clear that the operator $[\cdot]_{\mathcal{B}}$ is a linear transformation, i.e.,

$$[p\boldsymbol{v} + q\boldsymbol{w}]_{\mathcal{B}} = p[\boldsymbol{v}]_{\mathcal{B}} + q[\boldsymbol{w}]_{\mathcal{B}} \quad \forall p, q \in \mathbb{F}$$

3. The operator $[\cdot]_{\mathcal{B}}$ is surjective:

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \implies \boldsymbol{v} = 0\boldsymbol{v}_1 + \cdots + 0\boldsymbol{v}_n = \mathbf{0}.$$

4. The injective is clear, i.e., $[\boldsymbol{v}]_{\mathcal{B}} = [\boldsymbol{w}]_{\mathcal{B}}$ implies $\boldsymbol{v} = \boldsymbol{w}$.

Therefore, $[\cdot]_{\mathcal{B}}$ is an isomorphism. ■

We can use the Theorem (3.1) to simplify computations in vector spaces:

■ **Example 3.2** Given a vector sapce $V = P_3[x]$ and its basis $B = \{1, x, x^2, x^3\}$.

To check if the set $\{1 + x^2, 3 - x^3, x - x^3\}$ is linearly independent, by part (1) in Proposition (3.1) and Theorem (3.1), it suffices to check whether the corresponding coordinate vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

is linearly independent, i.e., do Gaussian Elimination and check the number of pivots. ■

Here gives rise to the question: if $\mathcal{B}_1, \mathcal{B}_2$ form two basis of V, then how are $[\boldsymbol{v}]_{\mathcal{B}_1}, [\boldsymbol{v}]_{\mathcal{B}_2}$ related to each other?

Here we consider an easy example first:

■ **Example 3.3** Consider $V = \mathbb{R}^n$ and its basis $\mathcal{B}_1 = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$. For any $\boldsymbol{v} \in V$,

$$\boldsymbol{v} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \alpha_n \boldsymbol{e}_1 + \cdots + \alpha_n \boldsymbol{e}_n \implies [\boldsymbol{v}]_{\mathcal{B}_1} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Also, we can construct a different basis of V:

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\},$$

which gives a different coordinate vector of $\boldsymbol{v}$:

$$[\boldsymbol{v}]_{\mathcal{B}_2} = \begin{pmatrix} \alpha_1 - \alpha_2 \\ \alpha_2 - \alpha_3 \\ \vdots \\ \alpha_{n-1} - \alpha_n \\ \alpha_n \end{pmatrix}$$

■

Proposition 3.2 — Change of Basis. Let $\mathcal{A} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ and $\mathcal{A}' = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ be two ordered basis of a vector space V. Define the **change of basis** matrix from $\mathcal{A}$ to $\mathcal{A}'$, say $\mathcal{C}_{\mathcal{A}',\mathcal{A}} := [\alpha_{ij}]$, where

$$\boldsymbol{v}_j = \sum_{i=1}^{m} \alpha_{ij} \boldsymbol{w}_i$$

Then for any vector $\boldsymbol{v} \in V$, the *change of basis amounts to left-multiplying the change of basis matrix*:

$$\mathcal{C}_{\mathcal{A}',\mathcal{A}} [\boldsymbol{v}]_A = [\boldsymbol{v}]_{A'} \tag{3.1}$$

Define matrix $\mathcal{C}_{\mathcal{A},\mathcal{A}'} := [\beta_{ij}]$, where

$$\boldsymbol{w}_j = \sum_{i=1}^{n} \beta_{ij} \boldsymbol{v}_i$$

Then we imply that

$$(\mathcal{C}_{\mathcal{A},\mathcal{A}'})^{-1} = \mathcal{C}_{\mathcal{A}',\mathcal{A}}$$

Proof. 1. First show (3.1) holds for $\boldsymbol{v} = \boldsymbol{v}_j$, $j = 1, \dots, n$:

$$\text{LHS of (3.1)} = [\alpha_{ij}]\boldsymbol{e}_j = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix}$$

$$\text{RHS of (3.1)} = [\boldsymbol{v}_j]_{\mathcal{A}'} = \left[\sum_{i=1}^{n} \alpha_i \boldsymbol{w}_i \right]_{\mathcal{A}'} = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix}$$

Therefore,

$$\mathcal{C}_{\mathcal{A}',\mathcal{A}}[\boldsymbol{v}_j]_{\mathcal{A}} = [\boldsymbol{v}_j]_{\mathcal{A}'}, \quad \forall j = 1, \dots, n. \tag{3.2}$$

2. Then for any $\boldsymbol{v} \in V$, we imply $\boldsymbol{v} = r_1\boldsymbol{v}_1 + \cdots + r_n\boldsymbol{v}_n$, which implies that

$$\begin{aligned}
\mathcal{C}_{\mathcal{A}',\mathcal{A}}[\boldsymbol{v}]_{\mathcal{A}} &= \mathcal{C}_{\mathcal{A}',\mathcal{A}}[r_1\boldsymbol{v}_1 + \cdots + r_n\boldsymbol{v}_n]_{\mathcal{A}} && \text{(3.3a)} \\
&= \mathcal{C}_{\mathcal{A}',\mathcal{A}}\left(r_1[\boldsymbol{v}_1]_{\mathcal{A}} + \cdots + r_n[\boldsymbol{v}_n]_{\mathcal{A}}\right) && \text{(3.3b)} \\
&= \sum_{j=1}^{n} r_j \mathcal{C}_{\mathcal{A}',\mathcal{A}}[\boldsymbol{v}_j]_{\mathcal{A}} && \text{(3.3c)} \\
&= \sum_{j=1}^{n} r_j [\boldsymbol{v}_j]_{\mathcal{A}'} && \text{(3.3d)} \\
&= \left[\sum_{j=1}^{n} r_j \boldsymbol{v}_j \right]_{\mathcal{A}'} && \text{(3.3e)} \\
&= [\boldsymbol{v}]_{\mathcal{A}'} && \text{(3.3f)}
\end{aligned}$$

where (3.3a) and (3.3e) is by applying the lineaity of $[\cdot]_{\mathcal{A}}$ and $[\cdot]_{\mathcal{A}'}$; (3.3d) is by applying the result (3.12). Therefore (3.1) is shown for $\forall \boldsymbol{v} \in V$.

3. Now we show that $(\mathcal{C}_{\mathcal{A}\mathcal{A}'}\mathcal{C}_{\mathcal{A}'\mathcal{A}}) = \boldsymbol{I}_n$. Note that

$$\begin{aligned}\boldsymbol{v}_j &= \sum_{i=1}^{n} \alpha_{ij}\boldsymbol{w}_i \\ &= \sum_{i=1}^{n} \alpha_{ij} \sum_{k=1}^{n} \beta_{ki}\boldsymbol{v}_k \\ &= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \beta_{ki}\alpha_{ij} \right) \boldsymbol{v}_i\end{aligned}$$

By the uniqueness of coordinates, we imply

$$\left(\sum_{i=1}^{n} \beta_{ki}\alpha_{ij} \right) = \delta_{jk} := \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

By the matrix multiplication, the (k, j)-th entry for $\mathcal{C}_{\mathcal{A}\mathcal{A}'}\mathcal{C}_{\mathcal{A}'\mathcal{A}}$ is

$$[\mathcal{C}_{\mathcal{A}\mathcal{A}'}\mathcal{C}_{\mathcal{A}'\mathcal{A}}]_{kj} = \left(\sum_{i=1}^{n} \beta_{ki}\alpha_{ij} \right) = \delta_{jk} \implies (\mathcal{C}_{\mathcal{A}\mathcal{A}'}\mathcal{C}_{\mathcal{A}'\mathcal{A}}) = \boldsymbol{I}_n$$

Noew, suppose

$$\begin{aligned}\boldsymbol{v}_j &= \sum_{i=1}^{n} \alpha_{ij}\boldsymbol{w}_i \\ &= \sum_{i=1}^{n} \alpha_{ij} \sum_{k=1}^{n} \beta_{ki}\boldsymbol{v}_k \\ &= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \beta_{ki}\alpha_{ij} \right) \boldsymbol{v}_i\end{aligned}$$

By the uniqueness of coordinates, we imply

$$\left(\sum_{i=1}^{n} \beta_{ki}\alpha_{ij} \right) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

where

$$\left(\sum_{i=1}^{n} \beta_{ki}\alpha_{ij} \right) = (\mathcal{C}_{AA'}\mathcal{C}_{A'A}).$$

Therefore, $(\mathcal{C}_{AA'}\mathcal{C}_{A'A}) = \boldsymbol{I}_n$. ∎

■ **Example 3.4** Back to Example (3.3), write $\mathcal{B}_1, \mathcal{B}_2$ as

$$\mathcal{B}_1 = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}, \quad \mathcal{B}_2 = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$$

and therefore $\boldsymbol{w}_i = \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_i$. The change of basis matrix is given by

$$C_{\mathcal{B}_1,\mathcal{B}_2} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

which implies that for $\boldsymbol{v}$ in the example,

$$C_{\mathcal{B}_1,\mathcal{B}_2}[\boldsymbol{v}]_{\mathcal{B}_2} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 - \alpha_2 \\ \vdots \\ \alpha_{n-1} - \alpha_n \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = [\boldsymbol{v}]_{\mathcal{B}_1}$$

Definition 3.2 Let $T : V \to W$ be a linear transformation, and

$$\mathcal{A} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}, \quad \mathcal{B} = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\}$$

be basis of V and W, respectively. The **matrix representation** of T with respect to (w.r.t.) $\mathcal{A}$ and $\mathcal{B}$ is defined as $(T)_{\mathcal{B}\mathcal{A}} \in M_{m\times m}(\mathbb{F})$, where

$$T(\boldsymbol{v}_j) = \sum_{i=1}^{m} \alpha_{ij}\boldsymbol{w}_j$$