2.4. Wednesday for MAT3040

Reviewing.

- Basis, Dimension
- Basis Extension
- $W_1 \bigcap W_2 = \emptyset$ implies $W_1 \oplus W_2 = W_1 + W_2$ (Direct Sum).

2.4.1. Remark on Direct Sum

Proposition 2.13 The set $W_1 + W_2 = W_1 \oplus W_2$ iff any $\boldsymbol{w} \in W_1 + W_2$ can be uniquely expressed as

$$\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_2,$$

where $\boldsymbol{w}_i \in W_i$ for $i = 1,2$.

(R) We can also define addiction among finite set of vector spaces $\{W_1, \dots, W_k\}$.

If $\boldsymbol{w}_1 + \cdots + \boldsymbol{w}_k = \boldsymbol{0}$ implies $\boldsymbol{w}_i = 0, \forall i$, then we can write $W_1 + \cdots + W_k$ as

$$W_1 \oplus \cdots \oplus W_k$$

Proposition 2.14 — Complementation. Let $W \le V$ be a vector subspace of a fintie dimension vector space V. Then there exists $W' \le V$ such that

$$W \oplus W' = V.$$

Proof. It's clear that $\dim(W) := k \le n := \dim(V)$. Suppose $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is a basis of W.

By the basis extension proposition, we can extend it into $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_n\}$, which is a basis of V.

Therefore, we take $W' = \text{span}\{\boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_n\}$, which follows that

1. $W + W' = V$: $\forall \boldsymbol{v} \in V$ has the form

$$\boldsymbol{v} = (\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k) + (\alpha_{k+1} \boldsymbol{v}_{k+1} + \cdots + \alpha_n \boldsymbol{v}_n),$$

   where $\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k \in W$ and $\alpha_{k+1} \boldsymbol{v}_{k+1} + \cdots + \alpha_n \boldsymbol{v}_n \in W'$.
2. $W \bigcap W' = \{\mathbf{0}\}$: Suppose $\boldsymbol{v} \in W \bigcap W'$, i.e.,

$$\begin{aligned} \boldsymbol{v} &= (\beta_1 \boldsymbol{v}_1 + \cdots + \beta_k \boldsymbol{v}_k) + (0\boldsymbol{v}_{k+1} + \cdots + 0\boldsymbol{v}_n) \in W \\ &= (0\boldsymbol{v}_1 + \cdots + 0\boldsymbol{v}_k) + (\beta_{k+1} \boldsymbol{v}_{k+1} + \cdots + \beta_n \boldsymbol{v}_n) \in W'. \end{aligned}$$

   By the uniqueness of coordinates, we imply $\beta_1 = \cdots = \beta_n = 0$, i.e., $\boldsymbol{v} = \mathbf{0}$.

Therefore, we conclude that $W \oplus W' = V$. ■

2.4.2. Linear Transformation

Definition 2.7 [Linear Transformation] Let V, W be vector spaces. Then $T : V \to W$ is a **linear transformation** if

$$T(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2) = \alpha T(\boldsymbol{v}_1) + \beta T(\boldsymbol{v}_2),$$

for $\forall \alpha, \beta \in \mathbb{F}$ and $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$.

■ **Example 2.12**

1. The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined as $\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}$ (where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$) is a linear transformation.
2. The transformation $T : \mathbb{R}[x] \to \mathbb{R}[x]$ defined as

$$p(x) \mapsto T(p(x)) = p'(x), \quad p(x) \mapsto T(p(x)) = \int_0^x p(t)\,\mathrm{d}t$$

   is a linear transformation

3. The transformation $T : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ defined as

$$\boldsymbol{A} \mapsto \mathrm{trace}(\boldsymbol{A}) := \sum_{i=1}^{n} a_{ii}$$

is a linear transformation.

However, the transformation

$$\boldsymbol{A} \mapsto \det(\boldsymbol{A})$$

is not a linear transformation.

Definition 2.8 [Kernel/Image] Let $T : V \to W$ be a linear transfomation.

1. The **kernel** of T is

$$\ker(T) = T^{-1}(\mathbf{0}) = \{\boldsymbol{v} \in V \mid T(\boldsymbol{v}) = \mathbf{0}\}$$

2. The **image** (or range) of T is

$$\mathrm{Im}(T) = T(\boldsymbol{v}) = \{T(\boldsymbol{v}) \in W \mid \boldsymbol{v} \in V\}$$

Example 2.13

1. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$, then

$$\ker(T) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \mathbf{0}\} = \mathrm{Null}(\boldsymbol{A}) \qquad \text{Null Space}$$

and

$$\mathrm{Im}(T) = \{\boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{R}^n\} = \mathrm{Col}(\boldsymbol{A}) = \mathrm{span}\{\text{columns of } \boldsymbol{A}\} \qquad \text{Column Space}$$

2. For $T(p(x)) = p'(x)$, $\ker(T) = \{\text{constant polynomials}\}$ and $\mathrm{Im}(T) = \mathbb{R}[x]$.

Proposition 2.15 The kernel or image for a linear transformation $T : V \to W$ also forms a vector subspace:

$$\ker(T) \le V, \quad \mathrm{Im}(T) \le W$$

Proof. For $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \ker(T)$, we imply

$$T(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2) = \mathbf{0},$$

which implies $\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2 \in \ker(T)$.

The remaining proof follows similarly. ∎

Definition 2.9 [Rank/Nullity] Let V, W be finite dimensional vector spaces and $T : V \to W$ a linear transformation. Then we define

$$\mathrm{rank}(T) = \dim(\mathrm{im}(T))$$
$$\mathrm{nullity}(T) = \dim(\ker(T))$$

R Let

$$\mathrm{Hom}_{\mathbb{F}}(V, W) = \{\text{all linear transformations } T : V \to W\},$$

and we can define the addiction and scalar multiplication to make it a vector space:

1. For $T, S \in \mathrm{Hom}_{\mathbb{F}}(V, W)$, define

   $$(T + S)(\boldsymbol{v}) = T(\boldsymbol{v}) + S(\boldsymbol{v}),$$

   which implies $T + S \in \mathrm{Hom}_{\mathbb{F}}(V, W)$.
2. Also, define

   $$(\gamma T)(\boldsymbol{v}) = \gamma T(\boldsymbol{v}), \qquad \text{for } \forall \gamma \in \mathbb{F},$$

   which implies $\gamma T \in \mathrm{Hom}_{\mathbb{F}}(V, W)$.

In particular, if $V = \mathbb{R}^n, W = \mathbb{R}^m$, then

$$\mathrm{Hom}_{\mathbb{F}}(V,W) = M_{m\times n}(\mathbb{R}).$$

Proposition 2.16 If $\dim(V) = n, \dim(W) = m$, then $\dim(\mathrm{Hom}_{\mathbb{F}}(V,W)) = mn$.

Proposition 2.17 There are anternative characterizations for the injectivity and surjectivity of lienar transformation T:

1. The linear transformation T is injective if and only if

$$\ker(T) = 0, \Longleftrightarrow \mathrm{nullity}(T) = 0.$$

2. The linear transformation T is surjective if and only if

$$\mathrm{im}(T) = W, \Longleftrightarrow \mathrm{rank}(T) = \dim(W).$$

3. If T is bijective, then T^{-1} is a linear transformation.

Proof. 1. (a) For the forward direction of (1),

$$\boldsymbol{x} \in \ker(T) \implies T(\boldsymbol{x}) = 0 = T(\mathbf{0}) \implies \boldsymbol{x} = \mathbf{0}$$

(b) For the reverse direction of (1),

$$T(\boldsymbol{x}) = T(\boldsymbol{y}) \implies T(\boldsymbol{x} - \boldsymbol{y}) = \mathbf{0} \implies \boldsymbol{x} - \boldsymbol{y} \in \ker(T) = \mathbf{0} \implies \boldsymbol{x} = \boldsymbol{y}$$

2. The proof follows similar idea in (1).
3. Let $T^{-1} : W \to V$. For all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in W$, there exists $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ such that $T(\boldsymbol{v}_i) = \boldsymbol{w}_i$, i.e., $T^{-1}(\boldsymbol{w}_i) = \boldsymbol{v}_i$ $i = 1,2$.

Consider the mapping

$$
\begin{aligned}
T(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2) &= \alpha T(\boldsymbol{v}_1) + \beta T(\boldsymbol{v}_2) \\
&= \alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2,
\end{aligned}
$$

which implies $\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2 = T^{-1}(\alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2)$, i.e.,

$$
\alpha T^{-1}(\boldsymbol{w}_1) + \beta T^{-1}(\boldsymbol{w}_2) = T^{-1}(\alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2).
$$

■

Definition 2.10 [isomorphism]

We say the vector subspaces V and W are isomorphic if there exists a bijective linear transfomation $T : V \to W$. ($V \cong W$)

This mapping T is called an **isomorphism** from V to W.

R If $\dim(V) = \dim(W) = n < \infty$, then $V \cong W$:

Take $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}, \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ as basis of V and W, respectively. Then one can construct $T : V \to W$ satisfying $T(\boldsymbol{v}_i) = \boldsymbol{w}_i$ for $\forall i$ as follows:

$$
T(\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n) = \alpha_n \boldsymbol{w}_1 + \cdots + \alpha_n \boldsymbol{w}_n \ \forall \alpha_i \in \mathbb{F}
$$

It's clear that our constructed T is a linear transformation.

R $V \cong W$ doesn't imply any linear transformations $T : V \to W$ is an isomorphism. e.g., $T(\boldsymbol{v}) = \boldsymbol{0}$ is not an isomorphic if $W \neq \{\boldsymbol{0}\}$.

Theorem 2.3 — Rank-Nullity Theorem. Let $T : V \to W$ be a linear transformation with $\dim(V) < \infty$. Then

$$
\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V).
$$

Proof. Since $\ker(T) \leq V$, by proposition (2.14), there exists $V_1 \leq V$ such that

$$V = \ker(T) \oplus V_1.$$

1. Consider the transformation $T\,|_{V_1} \colon V_1 \to T(V_1)$, which is an isomorphism, since:
   - Surjectivity is immediate
   - For $\boldsymbol{v} \in \ker(T\,|_{V_1})$,

     $$T(\boldsymbol{v}) = \mathbf{0} \implies \boldsymbol{v} \in \ker(T),$$

     which implies $\boldsymbol{v} = \mathbf{0}$ since $\boldsymbol{v} \in \ker(T) \cap V_1 = 0$, i.e., the injectivity follows.

   Therefore, $\dim(V_1) = \dim(T(V_1))$.
2. Secondly, given an isomorphism T from X to Y with $\dim(X) < \infty$, then $\dim(X) = \dim(T(X))$. The reason follows from assignment 1 questions (8-9):

   $$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \text{ is a basis of } X \implies \{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_k)\} \text{ is a basis of } Y$$
3. Note that $T(V_1) = T(V) = \mathrm{im}(T)$, since:
   - for $\forall \boldsymbol{v} \in V$, $\boldsymbol{v} = \boldsymbol{v}_k + \boldsymbol{v}_1$, where $\boldsymbol{v}_k \in \ker(T), \boldsymbol{v}_1 \in V_1$, which implies

     $$T(\boldsymbol{v}) = T(\boldsymbol{v}_k) + T(\boldsymbol{v}_1) = \mathbf{0} + T(\boldsymbol{v}_1),$$

     i.e., $T(V) \subseteq T(V_1) \subseteq T(V)$, i.e., $T(V) = T(V_1)$.
4. By the proof of complementation,

$$\begin{aligned}
\dim(V) &= \dim(\ker(T)) + \dim(V_1) \\
&= \mathrm{nullity}(T) + \dim(T(V_1)) \\
&= \mathrm{nullity}(T) + \dim(T(V)) \\
&= \mathrm{nullity}(T) + \dim(\mathrm{im}(T)) \\
&= \mathrm{nullity}(T) + \mathrm{rank}(T).
\end{aligned}$$