Chapter 2

Week2

2.1. Monday for MAT3040

Reviewing.

1. Linear Combination and Span
2. Linear Independence
3. Basis: a set of vectors $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\}$ is called a **basis** for V if $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\}$ is linearly independent, and $V = \text{span}\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\}$.

   Lemma: Given $V = \text{span}\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\}$, we can find a basis for this set. Here V is said to be **finitely generated**.
4. Lemma: The vector $\boldsymbol{w} \in \text{span}\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_n\} \setminus \text{span}\{\boldsymbol{v}_2,\dots,\boldsymbol{v}_n\}$ implies that

$$\boldsymbol{v}_1 \in \text{span}\{\boldsymbol{w},\boldsymbol{v}_2,\dots,\boldsymbol{v}_n\} \setminus \text{span}\{\boldsymbol{v}_2,\dots,\boldsymbol{v}_n\}$$

2.1.1. Basis and Dimension

Theorem 2.1 Let V be a finitely generated vector space. Suppose $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_m\}$ and $\{\boldsymbol{w}_1,\dots,\boldsymbol{w}_n\}$ are two basis of V. Then $m = n$. (where m is called the **dimension**)

Proof. Suppose on the contrary that $m \neq n$. Without loss of generality (w.l.o.g.), assume that $m < n$. Let $\boldsymbol{v}_1 = \alpha_1\boldsymbol{w}_1 + \cdots + \alpha_n\boldsymbol{w}_n$, with some $\alpha_i \neq 0$. w.l.o.g., assume $\alpha_1 \neq 0$. Therefore,

$$\boldsymbol{v}_1 \in \text{span}\{\boldsymbol{w}_1,\boldsymbol{w}_2,\dots,\boldsymbol{w}_n\} \setminus \text{span}\{\boldsymbol{w}_2,\dots,\boldsymbol{w}_n\} \tag{2.1}$$

which implies that $\boldsymbol{w}_1 \in \text{span}\{\boldsymbol{v}_1,\boldsymbol{w}_2,\dots,\boldsymbol{w}_n\} \setminus \text{span}\{\boldsymbol{w}_2,\dots,\boldsymbol{w}_n\}$.

Then we claim that $\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$ is a basis of V:

1. Note that $\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$ is a spannning set:

$$\begin{aligned}\boldsymbol{w}_1 \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} &\implies \{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} \subseteq \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} \\ &\implies \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} \subseteq \operatorname{span}\{\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}\} \subseteq \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}\end{aligned}$$

   Since $V = \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$, we have $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} = V$.

2. Then we show the linear independence of $\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$. Consider the equation

$$\beta_1 \boldsymbol{v}_1 + \beta_2 \boldsymbol{v}_2 + \cdots + \beta_n \boldsymbol{w}_n = \mathbf{0}$$

   (a) When $\beta_1 \neq 0$, we imply

$$\boldsymbol{v}_1 = \left(-\frac{\beta_2}{\beta_1}\right) \boldsymbol{w}_2 + \cdots + \left(-\frac{\beta_n}{\beta_1}\right) \boldsymbol{w}_n \in \operatorname{span}\{\boldsymbol{w}_2, \dots, \boldsymbol{w}_n\},$$

   which contradicts (2.1).

   (b) When $\beta_1 = 0$, then $\beta_2 \boldsymbol{w}_2 + \cdots + \beta_n \boldsymbol{w}_n = \mathbf{0}$, which implies $\beta_2 = \cdots = \beta_n = 0$, due to the independence of $\{\boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$.

Therefore, $\boldsymbol{v}_2 \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$, i.e.,

$$\boldsymbol{v}_2 = \gamma_1 \boldsymbol{v}_1 + \cdots + \gamma_n \boldsymbol{v}_n,$$

where $\gamma_2, \dots, \gamma_n$ cannot be all zeros, since otherwise $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ are linearly dependent, i.e., $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ cannot form a basis. w.l.o.g., assume $\gamma_2 \neq 0$, which implies

$$\boldsymbol{w}_2 \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}_3, \dots, \boldsymbol{w}_n\} \setminus \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_3, \dots, \boldsymbol{w}_n\}.$$

Following the simlar argument above, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}_3, \dots, \boldsymbol{w}_n\}$ forms a basis of V.

Continuing the argument above, we imply $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m, \boldsymbol{w}_{m+1}, \dots, \boldsymbol{w}_n\}$ is a basis of V.

Since $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ is a basis as well, we imply

$$\boldsymbol{w}_{m+1} = \delta_1 \boldsymbol{v}_1 + \cdots + \delta_m \boldsymbol{v}_m$$

for some $\delta_i \in \mathbb{F}$, i.e., $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m, \boldsymbol{w}_{m+1}\}$ is linearly dependent, which is a contradction.

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■ **Example 2.1** A vector space may have more than one basis.

Suppose $V = \mathbb{F}^n$, it is clear that $\dim(V) = n$, and

$\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is a basis of V, where $\boldsymbol{e}_i$ denotes a unit vector.

There could be other basis of V, such as

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \right\}$$

Actually, the columns of any invertible $n \times n$ matrix forms a basis of V. ■

■ **Example 2.2** Suppose $V = M_{m \times n}(\mathbb{R})$, we claim that $\dim(V) = mn$:

$$\left\{ E_{ij} \left| \begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix} \right. \right\} \text{ is a basis of } V,$$

where E_{ij} is $m \times n$ matrix with 1 at (i,j)-th entry, and 0s at the remaining entries. ■

■ **Example 2.3** Suppose $V = \{\text{all polynomials of degree} \le \text{n}\}$, then $\dim(V) = n + 1$. ■

■ **Example 2.4** Supppose $V = \{\boldsymbol{A} \in M_{n \times n}(\mathbb{R}) \mid \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}\}$, then $\dim(V) = \frac{n(n+1)}{2}$. ■

■ **Example 2.5** Let $W = \{\boldsymbol{B} \in M_{n\times n}(\mathbb{R}) \mid \boldsymbol{B}^{\mathrm{T}} = -\boldsymbol{B}\}$, then $\dim(V) = \frac{n(n-1)}{2}$. ■

(R) Sometimes it should be classified the field $\mathbb{F}$ for the scalar multiplication to define a vector space. Conside the example below:

1. Let $V = \mathbb{C}$, then $\dim(\mathbb{C}) = 1$ for the scalar multiplication defined under the field $\mathbb{C}$.
2. Let $V = \text{span}\{1, i\} = \mathbb{C}$, then $\dim(\mathbb{C}) = 2$ for the scalar multiplication defined under the field $\mathbb{R}$, since all $z \in V$ can be written as $z = a + bi$, $\forall a, b \in \mathbb{R}$.
3. Therefore, to aviod confusion, it is safe to write

$$\dim_{\mathbb{C}}(\mathbb{C}) = 1, \quad \dim_{\mathbb{R}}(\mathbb{C}) = 2.$$

2.1.2. Operations on a vector space

Note that the basis for a vector space is characterized as the **maximal linearly independent set**.

Theorem 2.2 — **Basis Extension.** Let V be a finite dimensional vector space, and $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ be a linearly independent set on V, Then we can extend it to the basis $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_n\}$ of V.

Proof.

- Suppose $\dim(V) = n > k$, and $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ is a basis of V. Consider the set $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\} \bigcup \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$, which is linearly dependent, i.e.,

$$\alpha_1 \boldsymbol{w}_1 + \dots + \alpha_n \boldsymbol{w}_n + \beta_1 \boldsymbol{v}_1 + \dots + \beta_k \boldsymbol{v}_k = \boldsymbol{0},$$

with some $\alpha_i \neq 0$, since otherwise this equation will only have trivial solution. w.l.o.g., assume $\alpha_1 \neq 0$.

- Therefore, consider the set $\{\boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} \bigcup \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$. We keep removing elements from $\{\boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$ until we first get the set

$$S \bigcup \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\},$$

with $S \subseteq \{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$ and $S \bigcup \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is linearly independent, i.e., S is a maximal subset of $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ such that $S \bigcup \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is linearly independent.

- Rewrite $S = \{\boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_m\}$ and therefore $S' = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_m\}$ are linearly independent. It suffices to show S' spans V.
  - Indeed, for all $\boldsymbol{w}_i \in \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$, $\boldsymbol{w}_i \in \text{span}(S')$, since otherwise the equation

    $$\alpha \boldsymbol{w}_i + \beta_1 \boldsymbol{v}_1 + \cdots + \beta_m \boldsymbol{v}_m = \mathbf{0} \implies \alpha = 0,$$

    which implies that $\beta_1 \boldsymbol{v}_1 + \cdots + \beta_m \boldsymbol{v}_m = \mathbf{0}$ admits only trivial solution, i.e.,

    $$\{\boldsymbol{w}_i\} \bigcup S' = \{\boldsymbol{w}_i\} \bigcup S \bigcup \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\} \text{ is linearly independent,}$$

    which violetes the maximality of S.

  Therefore, all $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\} \subseteq \text{span}(S')$, which implies $\text{span}(S') = V$.

  Therefore, S' is a basis of V.

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(R) Start with a spanning set, we keep removing something to form a basis; start with independent set, we keep adding something to form a basis.

In other words, the basis is both the minimal spanning set, and the maximal linearly independent set.

Definition 2.1 [Direct Sum] Let W_1, W_2 be two vector subspaces of V, then

1. $W_1 \bigcap W_2 := \{\boldsymbol{w} \in V \mid \boldsymbol{w} \in W_1, \text{ and } \boldsymbol{w} \in W_2\}$
2. $W_1 + W_2 := \{\boldsymbol{w}_1 + \boldsymbol{w}_2 \mid \boldsymbol{w}_i \in W_i\}$

3. If furthermore that $W_1 \bigcap W_2 = \{\mathbf{0}\}$, then $W_1 + W_2$ is denoted as $W_1 \oplus W_2$, which is called **direct sum**.

Proposition 2.1 $W_1 \bigcap W_2$ and $W_1 + W_2$ are vector subspaces of V.