Chapter 14

Week14

14.1. Monday for MAT3040

14.1.1. Multilinear Tensor Product

Definition 14.1 [Tensor Product among More spaces] Let $V_1,\dots,V_p$ be vector spaces over $\mathbb{F}$. Let $S = \{(v_1,\dots,v_p) \mid v_i \in V_i\}$ (We assume no relations among distinct elements in S), and define $\mathfrak{X} = \text{span}(S)$.

1. Then define the tensor product space $V_1 \otimes \cdots \otimes V_p = \mathfrak{X}/y$, where y is the vector subspace of $\mathfrak{X}$ spanned by vectors of the form

$$(v_1,\dots,v_i + v_i',\dots,v_p) - (v_1,\dots,v_i,\dots,v_p) - (v_1,\dots,v_i',\dots,v_p),$$

   and

$$(v_1,\dots,\alpha v_i,\dots,v_p) - \alpha(v_1,\dots,v_i,\dots,v_p)$$

   where $i = 1,2,\dots,p$.

2. The tensor product for vectors is defined as

$$v_1 \otimes \cdots \otimes v_p := \{(v_1,\dots,v_p) + y\} \in V_1 \otimes \cdots \otimes V_p$$

(R) Similar as in tensor product among two space,

1. We have

$$v_1 \otimes \cdots \otimes (\alpha v_i + \beta v_i') \otimes \cdots \otimes v_p = \alpha(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_p) + \beta(v_1 \otimes \cdots \otimes v_i' \otimes \cdots \otimes v_p)$$

2. A general vector in $V_1 \otimes \cdots \otimes V_p$ is

$$\sum_{i=1}^{n}(W_1^{(i)} \otimes \cdots \otimes W_p^{(i)}), \quad \text{where } W_j^{(i)} \in V_j, j = 1, \dots, p$$

3. Let $\mathcal{B}_i = \{v_i^{(1)}, \dots, v_i^{(\dim(V_i))}\}$ be a basis of $V_i, i = 1, \dots, p$, then

$$\mathcal{B} = \{V_1^{(\alpha_1)} \otimes \cdots \otimes V_p^{(\alpha_p)} \mid 1 \le \alpha_i \le \dim(V_i)\}$$

is a basis of $V_1 \otimes \cdots \otimes V_p$. As a result,

$$\dim(V_1 \otimes \cdots \otimes V_p) = (\dim(V_1)) \times \cdots \times (\dim(V_p))$$

Theorem 14.1 — Universal Property of multi-linear tensor. Let $\text{Obj} = \{\phi : V_1 \times \cdots \times V_p \to W \mid \phi \text{ is a } p\text{-linear map}\}$, i.e.,

$$\phi(v_1, \dots, \alpha v_i + \beta v_i', \dots, v_o) = \alpha\phi(v_1, \dots, v_i, \dots, v_p) + \beta\phi(v_1, \dots, v_i', \dots, v_p),$$
$$\forall v_i, v_i' \in V_i, i = 1, \dots, p, \forall \alpha, \beta \in \mathbb{F}.$$

For instance, the multiplication of p matrices is a p-linear map.

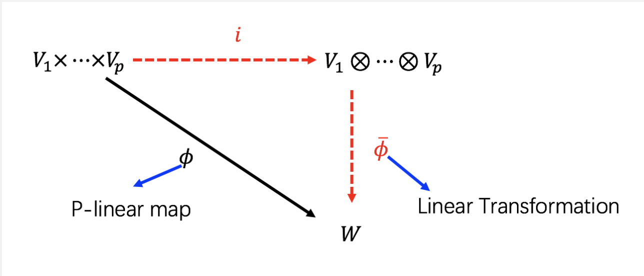
Then the mapping in the Obj,

$$\begin{aligned} i: \quad & V_1 \times V_p \to V_1 \otimes \cdots \otimes V_p \\ \text{with} \quad & (v_1, \dots, v_p) \mapsto v_1 \otimes \cdots \otimes v_p \end{aligned}$$

satisfies the universal property. In other words, for any $\phi : V_1 \times \cdots \times V_p \in \text{Obj}$, there exists the unqiue linear transformation

$$\bar{\phi} : V_1 \otimes \cdots \otimes V_p \to W$$

such that the diagram below commutes:



In other words, $\phi = \bar{\phi} \circ i$.

Corollary 14.1 Let $T_i : V_i \to V_i'$ be a linear transformation, $1 \le i \le p$. There is a unique linear transformation

$$\begin{aligned} (T_1 \otimes \cdots \otimes T_p) : \quad & V_1 \otimes \cdots \otimes V_p \to V_1' \otimes \cdots \otimes V_p' \\ \text{satisfying} \quad & (T_1 \otimes \cdots \otimes T_p)(v_1 \otimes \cdots \otimes v_p) = T_1(v_1) \otimes \cdots \otimes T_p(v_p) \end{aligned}$$

Proof. Construct the mapping

$$\begin{aligned} \phi : \quad & V_1 \times \cdots \times V_p \to V_1' \otimes \cdots \otimes V_p' \\ \text{with} \quad & (v_1, \dots, v_p) \mapsto T_1(v_1) \otimes \cdots \otimes T_p(v_p) \end{aligned}$$

which is indeed p-linear.

By the universal property, we induce the unique linear transformation

$$\bar{\phi} : V_1 \otimes \cdots \otimes V_p \to V_1' \otimes \cdots \otimes V_p'$$

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Notation. To make life easier, from now on, we only consider $V_1 = \cdots = V_p = V$. Then for any linear transformation $T: V \to W$, we have

$$T^{\otimes p}: V \otimes \cdots \otimes V \to W \otimes \cdots \otimes W$$

We use the short-hand notation $V^{\otimes p}$ to denote $\underbrace{V \otimes \cdots \otimes V}_{p \text{ terms in total}}$

Final Exam Ends Here.

14.1.2. Exterior Power

Definition 14.2 A p-linear map $\phi: V \times \cdots \times V \to W$ is called **alternating** if

$$\phi(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_p) = \mathbf{0}_W, \quad \text{provided that there exists some } v_i = v_j \text{ for } i \neq j.$$

Also, we say ϕ is p-alternating

Example 14.1 1. The cross product mapping

$$\begin{aligned} \phi: \quad & \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \\ \text{with} \quad & (\boldsymbol{v}, \boldsymbol{w}) \mapsto \boldsymbol{v} \times \boldsymbol{w} \end{aligned}$$

is alternating:

- ϕ is bilinear
- $\phi(\boldsymbol{v}, \boldsymbol{v}) = \boldsymbol{v} \times \boldsymbol{v} = \mathbf{0}$.

2. The determinant mapping

$$\begin{aligned} \phi: \quad & \underbrace{\mathbb{F}^n \times \cdots \times \mathbb{F}^n}_{n \text{ terms in total}} \to \mathbb{F} \\ \text{with} \quad & (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) \mapsto \det([\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n]) \end{aligned}$$

is alternating:

- ϕ is n-linear by MAT2040 knowledge
- ϕ is alternating by MAT2040 knowledge

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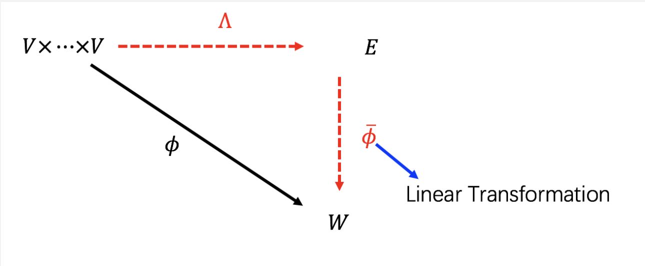
Theorem 14.2 — **Universal Property for exterior power.** Let $\text{Obj} := \{\phi : \underbrace{V \times \cdots V}_{p \text{ terms}} \to W \mid \phi \text{ is } p\text{-alternating map}\}$. Then there exists $\{\Lambda : V \times \cdots \times V \to E\} \in \text{Obj}$ satisfying the following:

- For all $\phi : V \times \cdots \times V \to W \in \text{Obj}$, there exists unique linear transformation $\bar{\phi} : E \to W$ satisfying



In other words, $\phi = \bar{\phi} \circ \Lambda$.