13.4. Wednesday for MAT3040

13.4.1. Tensor Product for Linear Transformations

Proposition 13.3 Suppose that $T : V \to V'$ and $S : W \to W'$ are linear transformations, then there exists an unique linear transformation

$$\begin{aligned} T \otimes S : \quad & V \otimes W \to V' \otimes W' \\ \text{satisfying} \quad & (T \otimes S)(v \otimes w) = T(v) \otimes S(w) \end{aligned}$$

Proof. We construct the mapping

$$\begin{aligned} T \times S : \quad & V \times W \to V' \otimes W' \\ \text{with} \quad & (T \times S)(v, w) = T(v) \otimes S(w) \end{aligned}$$

This mapping is indeed bilinear: for instance, we can show that

$$(T \times S)(av_1 + bv_2, w) = a(T \times S)(v_1, w) + b(T \times S)(v_2, w)$$

Therefore, $T \times S \in \text{Obj}$. Since the tensor product satisfies the universal property, we imply there exists an unique linear transformation

$$\begin{aligned} T \otimes S \quad & V \otimes W \to V' \otimes W' \\ \text{satisfying} \quad & (T \otimes S)(v \otimes w) = T(v) \otimes S(w) \end{aligned}$$

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Notation Warning. Does the notion $T \otimes S$ really form a tensor product, i.e., do we obtain the addictive rules for tensor product such as

$$(aT_1 + bT_2) \otimes S = a(T_1 \otimes S) + b(T_2 \otimes S)?$$

■ **Example 13.2** Let $V = V' = \mathbb{F}^2$ and $W = W' = \mathbb{F}^3$. Define the matrix-multiply mappings:

$$\begin{cases} T: & V \to V \\ \text{with} & \boldsymbol{v} \mapsto \boldsymbol{A}\boldsymbol{v} \\ & \boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{cases} \qquad \begin{cases} S: & W \to W \\ \text{with} & \boldsymbol{w} \mapsto \boldsymbol{B}\boldsymbol{w} \\ & \boldsymbol{B} = \begin{pmatrix} p & q & r \\ s & t & u \\ v & w & x \end{pmatrix} \end{cases}$$

How does $T \otimes S : V \otimes W \to V \otimes W$ look like?

- Suppose $\{e_1, e_2\}, \{f_1, f_2, f_3\}$ are usual basis of V, W, respectively. Then the basis of $V \otimes W$ is given by:

$$C = \{e_1 \otimes f_1, e_1 \otimes f_2, e_1 \otimes f_3, e_2 \otimes f_1, e_2 \otimes f_2, e_2 \otimes f_3\}.$$

- As a result, we can compute $(T \otimes S)(e_i \otimes f_j)$ for $i = 1,2$ and $j = 1,2,3$. For instance,

$$\begin{aligned}(T \otimes S)(e_1 \otimes e_1) &= T(e_1) \otimes S(e_1) \\ &= (ae_1 + ce_2) \otimes (pe_1 + se_2 + ve_3) \\ &= (ap)e_1 \otimes e_1 + (as)e_1 \otimes e_2 + (av)e_1 \otimes e_3 + (cp)e_2 \otimes e_1 + (cs)e_2 \otimes e_2 + (cv)e_2 \otimes e_3\end{aligned}$$

- Therefore, we obtain a matrix representation for the linear transformation $(T \otimes S)$:

We want a matrix representation for $(T \otimes S)$:

$$(T \otimes S)_{C,C} = \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix},$$

which is a large matrix formed by taking all possible products between the elements of $\boldsymbol{A}$ and those of $\boldsymbol{B}$. This operation is called the **Kronecker Tensor Product**, see the command *kron* in MATLAB for detail. ■

Proposition 13.4 More generally, given the linear operator $T: V \to V$ and $S: W \to W$, let $\mathcal{A} = \{v_1, \dots, v_n\}, \mathcal{B} = \{w_1, \dots, w_m\}$ be a basis of V, W respectively, with

$$(T)_{\mathcal{A},\mathcal{A}} = (a_{ij}) \quad (S_{\mathcal{B},\mathcal{B}}) = (b_{ij}) := B$$

As a result, $(T \otimes S)_{C,C} = A \otimes B$, where $C = \{v_1 \otimes w_1, \dots, v_n \otimes w_m\}$, and $A \otimes B$ denotes the Kronecker tensor product, defined as the matrix

$$\begin{pmatrix} a_{1,1}B & \cdots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{n,1}B & \cdots & a_{n,n}B \end{pmatrix}.$$

Proof. Following the similar procedure as in Example (13.2) and applying the relation

$$\begin{aligned}(T \otimes S)(v_i \otimes w_j) &= T(v_i) \otimes S(w_j) \\ &= \left(\sum_{k=1}^{n} a_{ki} v_k\right) \otimes \left(\sum_{\ell=1}^{m} b_{\ell j} w_\ell\right) \\ &= \sum_{k=1}^{n} \sum_{\ell=1}^{m} (a_{ki} b_{\ell j}) v_k \otimes w_\ell\end{aligned}$$

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Proposition 13.5 The operation $T \otimes S$ satisfies all the properties of tensor product. For example,

$$(aT_1 + bT_2) \otimes S = a(T_1 \otimes S) + b(T_2 \otimes S)$$

$$T \otimes (cS_1 + dS_2) = c(T \otimes S_1) + d(T \otimes S_2)$$

Therefore, the usage of the notion "$\otimes$" is justified for the definition of $T \otimes S$.

Proof using matrix multiplication. For instance, consider the operation $(T + T') \otimes S$, with $(T)_{\mathcal{A},\mathcal{A}} = (a_{ij})$, $(T')_{\mathcal{A},\mathcal{A}} = (c_{ij}), (S)_{\mathcal{B},\mathcal{B}} = B$.

We compute its matrix representation directly:

$$\begin{aligned}((T+T')\otimes S)_{C,C} &= (T+T')_{\mathcal{A},\mathcal{A}}\otimes(S)_{\mathcal{B},\mathcal{B}}\\ &= [(T)_{\mathcal{A},\mathcal{A}}+(T')_{\mathcal{A},\mathcal{A}}]\otimes(S)_{\mathcal{B},\mathcal{B}}\\ &= (T)_{\mathcal{A},\mathcal{A}}\otimes(S)_{\mathcal{B},\mathcal{B}}+(T')_{\mathcal{A},\mathcal{A}}\otimes(S)_{\mathcal{B},\mathcal{B}}\end{aligned}$$

where the last equality is by the addictive rule for kronecker product for matrices. Therefore,

$$((T+T')\otimes S)_{C,C}=(T\otimes S)_{C,C}+(T'\otimes S)_{C,C}\implies (T+T')\otimes S=T\otimes S+T'\otimes S$$

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Proof using basis of $T\otimes S$. Another way of the proof is by computing

$$((T+T')\otimes S)(v_i\otimes w_j),$$

where $\{v_i\otimes w_j\mid 1\le i\le n, 1\le j\le m\}$ forms a basis of $(T+T')\otimes S$:

$$\begin{aligned}((T+T')\otimes S)(v_i\otimes w_j) &= (T+T')(v_i)\otimes S(w_j)\\ &= (T(v_i)+T'(v_i))\otimes S(w_j)\\ &= T(v_i)\otimes S(w_j)+T'(v_i)\otimes S(w_j)\\ &= (T\otimes S)(v_i\otimes w_j)+(T'\otimes S)(v_i\otimes w_j)\end{aligned}$$

Since $((T+T')\otimes S)(v_i\otimes w_j)$ coincides with $(T\otimes S+T'\otimes S)(v_i\otimes w_j)$ for all basis vectors $v_i\otimes w_j\in C$, we imply

$$(T+T')\otimes S=T\otimes S+T'\otimes S$$

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Proposition 13.6 Let A,C be linear operators from V to V, and B,D be linear operators from W to W, then

$$(A\otimes B)\circ(C\otimes D)=(AC)\otimes(BD)$$

Proposition 13.7 Define linear operators $A : V \to V$ and $B : W \to W$ with $\dim(V), \dim(W) < \infty$. Then

$$\det(A \otimes B) = (\det(A))^{\dim(W)} (\det(B))^{\dim(V)}$$

Corollary 13.3 There exists a linear transformation

$$\begin{aligned} \Phi : \quad & \mathrm{Hom}(V,V) \otimes \mathrm{Hom}(W,W) \to \mathrm{Hom}(V \otimes W, V \otimes W) \\ \text{with} \quad & A \otimes B \mapsto A \otimes B \end{aligned}$$

where the input of Φ is the tensor product of linear transformations, and the output is the linear transformation.

Proof. Construct the mapping

$$\begin{aligned} \Phi \quad & : \mathrm{Hom}(V,V) \times \mathrm{Hom}(W,W) \to \mathrm{Hom}(V \otimes W, V \otimes W) \\ \text{with} \quad & \Phi(A,B) = A \otimes B \end{aligned}$$

The Φ is indeed bilinear: for instance,

$$\begin{aligned} \Phi(pA + qC, B) &= (pA + qC) \otimes B \\ &= p(A \otimes B) + q(C \otimes B) \\ &= p\Phi(A,B) + q\Phi(C,B) \end{aligned}$$

This corollary follows from the universal property of tensor product. ■

R If assuming that $\dim(V), \dim(W) < \infty$, we imply

$$\begin{aligned} \dim(\text{Input space of } \Phi) &= \dim(\mathrm{Hom}(V,V))\dim(\mathrm{Hom}(W,W)) \\ &= [\dim(V)\dim(V)] \cdot [\dim(W)\dim(W)] = [\dim(V)\dim(W)]^2 \\ &= [\dim(V \otimes W)]^2 \\ &= \dim(\mathrm{Hom}(V \otimes W, V \otimes W)) \\ &= \dim(\text{Output space of } \Phi) \end{aligned}$$

Therefore, is Φ is an isomorphism? If so, then every linear operator $\alpha : V \otimes W \to V \otimes W$ can be expressed as

$$\alpha = A_1 \otimes B_1 + \cdots + A_k \otimes B_k$$

where $A_i : V \to V$ and $B_j : W \to W$.