Chapter 13

Week13

13.1. Monday for MAT3040

Reviewing.

1. Define $S = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \boldsymbol{v} \in V, \boldsymbol{w} \in W\}$ and $\mathfrak{X} = \text{span}(S)$. In $\mathfrak{X}$, there are no relations between distinct elements of S, e.g.,

$$2(\boldsymbol{v}, 0) + 3(0, \boldsymbol{w}) \neq 1(2\boldsymbol{v}, 3\boldsymbol{w})$$

   General element in $\mathfrak{X}$:

$$a_1(\boldsymbol{v}_1, \boldsymbol{w}_1) + \cdots + a_n(\boldsymbol{v}_n, \boldsymbol{w}_n),$$

   where $(\boldsymbol{v}_i, \boldsymbol{w}_i)$ are distinct.

2. Define the space $V \otimes W = \mathfrak{X}/y$, with

$$\boldsymbol{v} \otimes \boldsymbol{w} = 1(\boldsymbol{v}, \boldsymbol{w}) + y \in V \otimes W.$$

   General element in $\mathfrak{X}/y := V \otimes W$:

$$\begin{aligned} a_1(\boldsymbol{v}_1, \boldsymbol{w}_1) + \cdots + a_n(\boldsymbol{v}_n, \boldsymbol{w}_n) + y &= a_1((\boldsymbol{v}_1, \boldsymbol{w}_1) + y) + \cdots + a_n((\boldsymbol{v}_n, \boldsymbol{w}_n) + y) \\ &= a_1(\boldsymbol{v}_1 \otimes \boldsymbol{w}_1) + \cdots + a_n(\boldsymbol{v}_n \otimes \boldsymbol{w}_n) \\ &= (a_1\boldsymbol{v}_1) \otimes \boldsymbol{w}_1 + \cdots + (a_n\boldsymbol{v}_n) \otimes \boldsymbol{w}_n \end{aligned}$$

Therefore, a general element in $V \otimes W$ is of the form

$$\boldsymbol{v}_1' \otimes \boldsymbol{w}_1 + \cdots + \boldsymbol{v}_n' \otimes \boldsymbol{w}_n, \ \boldsymbol{v}_i' \in V, \boldsymbol{w}_i \in W. \tag{13.1}$$

Note that $V \otimes W$ is different from $V \times W$, where all elements in $V \times W$ can be expressed as $(\boldsymbol{v}, \boldsymbol{w})$.

3. The tensor product mapping

$$\begin{aligned} i: &\quad V \times W \to V \otimes W \\ \text{with} &\quad (\boldsymbol{v}, \boldsymbol{w}) \mapsto \boldsymbol{v} \otimes \boldsymbol{w} \end{aligned}$$

satisfies the universal property.

Here we present an example for computing tensor product by making use of the rules below:

$$\begin{aligned} (\boldsymbol{v}_1 + \boldsymbol{v}_2) \otimes \boldsymbol{w} &= \boldsymbol{v}_1 \otimes \boldsymbol{w} + \boldsymbol{v}_2 \otimes \boldsymbol{w} \\ \boldsymbol{v} \otimes (\boldsymbol{w}_1 + \boldsymbol{w}_2) &= (\boldsymbol{v} \otimes \boldsymbol{w}_1) + (\boldsymbol{v} \otimes \boldsymbol{w}_2) \\ (k\boldsymbol{v}) \otimes \boldsymbol{w} &= k(\boldsymbol{v} \otimes \boldsymbol{w}) \\ \boldsymbol{v} \otimes (k\boldsymbol{w}) &= k(\boldsymbol{v} \otimes \boldsymbol{w}) \end{aligned}$$

▪ **Example 13.1** Let $V = W = \mathbb{R}^2$, with

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Here we have

$$\begin{aligned}\begin{pmatrix}3\\1\end{pmatrix}\otimes\begin{pmatrix}-4\\2\end{pmatrix} &= (3\boldsymbol{e}_1+2\boldsymbol{e}_2)\otimes(-4\boldsymbol{e}_1+2\boldsymbol{e}_2)\\ &= (3\boldsymbol{e}_1)\otimes(-4\boldsymbol{e}_1+2\boldsymbol{e}_2)+(\boldsymbol{e}_2)\otimes(-4\boldsymbol{e}_1+2\boldsymbol{e}_2)\\ &= (3\boldsymbol{e}_1)\otimes(-4\boldsymbol{e}_1)+(3\boldsymbol{e}_1)\otimes(2\boldsymbol{e}_2)+(\boldsymbol{e}_2)\otimes(-4\boldsymbol{e}_1)+\boldsymbol{e}_2\otimes(2\boldsymbol{e}_2)\\ &= -12(\boldsymbol{e}_1\otimes\boldsymbol{e}_1)+6(\boldsymbol{e}_1\otimes\boldsymbol{e}_2)-4(\boldsymbol{e}_2\otimes\boldsymbol{e}_1)+2(\boldsymbol{e}_2\otimes\boldsymbol{e}_2)\end{aligned}$$

Exercise: Check that $\boldsymbol{e}_1\otimes\boldsymbol{e}_2+\boldsymbol{e}_2\otimes\boldsymbol{e}_1$ cannot be re-written as

$$(a\boldsymbol{e}_1+b\boldsymbol{e}_2)\otimes(c\boldsymbol{e}_1+d\boldsymbol{e}_2),\quad a,b,c,d\in\mathbb{R}.$$

13.1.1. Basis of $V\otimes W$

Motivation. Given that $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_n\}$ is a basis of V, and $\{\boldsymbol{w}_1,\dots,\boldsymbol{w}_m\}$ a basis of W, we aim to find a basis of $V\otimes W$ using $\boldsymbol{v}_i$'s and $\boldsymbol{w}_i$'s.

Proposition 13.1 The set $\{\boldsymbol{v}_i\otimes\boldsymbol{w}_j\mid 1\le i\le n, 1\le j\le m\}$ spans the tensor product space $V\otimes W$.

Proof. Consider any $\boldsymbol{v}\in V$ and $\boldsymbol{w}\in W$, and we want to express $\boldsymbol{v}\otimes\boldsymbol{w}$ in terms of $\boldsymbol{v}_i,\boldsymbol{w}_j$. Suppose that $\boldsymbol{v}=\alpha_1\boldsymbol{v}_1+\cdots+\alpha_n\boldsymbol{v}_n$ and $\boldsymbol{w}=\beta_1\boldsymbol{w}_1+\cdots+\beta_m\boldsymbol{w}_m$.

Substituting $\boldsymbol{v}=\alpha_1\boldsymbol{v}_1+\cdots+\alpha_n\boldsymbol{v}_n$ into the expression $\boldsymbol{v}\otimes\boldsymbol{w}$, we imply

$$\begin{aligned}\boldsymbol{v}\otimes\boldsymbol{w} &= (\alpha_1\boldsymbol{v}_1+\cdots+\alpha_n\boldsymbol{v}_n)\otimes\boldsymbol{w}\\ &= (\alpha_1\boldsymbol{v}_1)\otimes\boldsymbol{w}_1+\cdots+(\alpha_n\boldsymbol{v}_n)\otimes\boldsymbol{w}_n\\ &= \alpha_1(\boldsymbol{v}_1\otimes\boldsymbol{w})+\cdots+\alpha_n(\boldsymbol{v}_n\otimes\boldsymbol{w})\end{aligned}$$

For each $\boldsymbol{v}_i\otimes\boldsymbol{w}$, $i=1,\dots,n$, similarly,

$$\boldsymbol{v}_i\otimes\boldsymbol{w}=\beta_1(\boldsymbol{v}_i\otimes\boldsymbol{w}_1)+\cdots+\beta_m(\boldsymbol{v}_i\otimes\boldsymbol{w}_m).$$

Therefore,

$$\boldsymbol{v} \otimes \boldsymbol{w} = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j (\boldsymbol{v}_i \otimes \boldsymbol{w}_j) \tag{13.2}$$

By (13.1), any vector in $V \otimes W$ is of the form

$$\boldsymbol{v}^{(1)} \otimes \boldsymbol{w}^{(1)} + \cdots + \boldsymbol{v}^{(\ell)} \otimes \boldsymbol{w}^{(\ell)}$$

By (13.2), each $\boldsymbol{v}^{(k)} \otimes \boldsymbol{w}^{(k)}, k = 1, \ldots, \ell$, can be expressed as

$$\boldsymbol{v}^{(k)} \otimes \boldsymbol{w}^{(k)} = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i^{(k)} \beta_j^{(k)} (\boldsymbol{v}_i \otimes \boldsymbol{w}_j)$$

Therefore,

$$\boldsymbol{v}^{(1)} \otimes \boldsymbol{w}^{(1)} + \cdots + \boldsymbol{v}^{(\ell)} \otimes \boldsymbol{w}^{(\ell)} = \sum_{k=1}^{\ell} \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i^{(k)} \beta_j^{(k)} (\boldsymbol{v}_i \otimes \boldsymbol{w}_j)$$

In other words, $\{\boldsymbol{v}_i \otimes \boldsymbol{w}_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ spans $V \otimes W$. ■

Theorem 13.1 A basis of $V \otimes W$ is $\{\boldsymbol{v}_i \otimes \boldsymbol{w}_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$

Proof. By proposition (13.1), it suffices to show that the set $\{\boldsymbol{v}_i \otimes \boldsymbol{w}_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ is linear independent. Suppose that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} (\boldsymbol{v}_i \otimes \boldsymbol{w}_j) = \mathbf{0} \tag{13.3}$$

Suppose that $\{\phi_1, \ldots, \phi_n\}$ is a dual basis of V^*, and $\{\psi_1, \ldots, \psi_m\}$ is a dual basis of W^*. Construct the mapping

$$\pi_{p,q} : \quad V \times W \longrightarrow \mathbb{F}$$

$$\text{with} \quad \pi_{p,q} = \phi_p(\boldsymbol{v}) \psi_q(\boldsymbol{w})$$

- The mapping $\pi_{p,q}$ is actually bilinear: for instance,

$$\begin{aligned}\pi_{p,q}(a\boldsymbol{v}_1 + b\boldsymbol{v}_2, \boldsymbol{w}) &= \phi_p(a\boldsymbol{v}_1 + b\boldsymbol{v}_2)\psi_q(\boldsymbol{w}) \\ &= (a\phi_p(\boldsymbol{v}_1) + b\phi_p(\boldsymbol{v}_2))\psi_q(\boldsymbol{w}) \\ &= a\phi_p(\boldsymbol{v}_1)\psi_q(\boldsymbol{w}) + b\phi_p(\boldsymbol{v}_2)\psi_q(\boldsymbol{w}) \\ &= a\pi_{p,q}(\boldsymbol{v}_1, \boldsymbol{w}) + b\pi_{p,q}(\boldsymbol{v}_2, \boldsymbol{w}).\end{aligned}$$

  Following the similar ideas, we can check that $\pi_{p,q}(\boldsymbol{v}, a\boldsymbol{w}_1 + b\boldsymbol{w}_2) = a\pi_{p,q}(\boldsymbol{v}, \boldsymbol{w}_1) + b\pi_{p,q}(\boldsymbol{v}, \boldsymbol{w}_2)$.

- Therefore, $\pi_{p,q} \in \text{Obj}$. By the universal property of the tensor product, $\pi_{p,q}$ induces the unique linear transformation

$$\begin{aligned}\textstyle\prod_{p,q} : &\quad V \otimes W \to \mathbb{F} \\ \text{with} &\quad \textstyle\prod_{p,q}(\boldsymbol{v} \otimes \boldsymbol{w}) = \pi_{p,q}(\boldsymbol{v}, \boldsymbol{w})\end{aligned}$$

  In other words, $\prod_{p,q}(\boldsymbol{v} \otimes \boldsymbol{w}) = \phi_p(\boldsymbol{v})\psi_q(\boldsymbol{w})$.

- Applying the mapping $\Pi_{p,q}$ on both sides of (13.3), we imply

$$\Pi_{p,q}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}(\boldsymbol{v}_i \otimes \boldsymbol{w}_j)\right) = \Pi_{p,q}(\mathbf{0})$$

  Or equivalently,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}\Pi_{p,q}(\boldsymbol{v}_i \otimes \boldsymbol{w}_j) = 0,$$

  i.e.,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}\phi_p(\boldsymbol{v}_i)\psi_q(\boldsymbol{w}_j) = \alpha_{p,q} = 0$$

  Following this procedure, we can argue that $\alpha_{ij} = 0, \forall i, \forall j$.

■

Corollary 13.1 If $\dim(V), \dim(W) < \infty$, then $\dim(V \otimes W) = \dim(V)\dim(W)$

Proof. Check dimension of the basis of $V \otimes W$. ■

R The universal property can be very helpful. In particular, given a bilinear mapping, say $\phi : V \times W \to U$, we imply $\phi \in \text{Obj}$. By theorem (12.3), since i satisfies the universal property of tensor product, we can induce an unique linear transformation $\psi : V \otimes W \to U$.

Let's try another example for making use of the universal property:

Theorem 13.2 For finite dimension U and V,

$$V \otimes U \cong U \otimes V$$

Proof. Construct the mapping

$$\begin{aligned} \phi : \quad & V \times U \to U \otimes V \\ \text{with} \quad & \phi(\boldsymbol{v}, \boldsymbol{u}) = \boldsymbol{u} \otimes \boldsymbol{v} \end{aligned}$$

Indeed, ϕ is bilinear: for instance,

$$\begin{aligned} \phi(a\boldsymbol{v}_1 + b\boldsymbol{v}_2, \boldsymbol{u}) &= u \otimes (a\boldsymbol{v}_1 + b\boldsymbol{v}_2) \\ &= a(\boldsymbol{u} \otimes \boldsymbol{v}_1) + b(u \otimes \boldsymbol{v}_2) \\ &= a\phi(\boldsymbol{v}_1, \boldsymbol{u}) + b\phi(\boldsymbol{v}_2, \boldsymbol{u}) \end{aligned}$$

Therefore, $\phi \in \text{Obj}$. By the universal property of tensor product, we induce an unique linear transformation

$$\begin{aligned} \Phi : \quad & V \otimes U \to U \otimes V \\ \text{with} \quad & \Phi(\boldsymbol{v} \otimes \boldsymbol{u}) = \boldsymbol{u} \otimes \boldsymbol{v} \end{aligned}$$

Similarly, we may induce the linear transformation

$$\begin{aligned} \Psi : \quad & U \otimes V \to V \otimes U \\ \text{with} \quad & \Psi(\boldsymbol{u} \otimes \boldsymbol{v}) = \boldsymbol{v} \otimes \boldsymbol{u} \end{aligned}$$

Given any $\sum_i \boldsymbol{u}_i \otimes \boldsymbol{v}_i \in U \otimes V$, observe that

$$\begin{aligned}(\Phi \circ \Psi)\left(\sum_i \boldsymbol{u}_i \otimes \boldsymbol{v}_i\right) &= \Phi\left(\sum_i \Psi(\boldsymbol{u}_i \otimes \boldsymbol{v}_i)\right) \\ &= \Phi\left(\sum_i \boldsymbol{v}_i \otimes \boldsymbol{u}_i\right) \\ &= \sum_i \Phi(\boldsymbol{v}_i \otimes \boldsymbol{u}_i) \\ &= \sum_i \boldsymbol{u}_i \otimes \boldsymbol{v}_i\end{aligned}$$

Therefore, $\Phi \circ \Psi = \mathrm{id}_{U \otimes V}$. Similarly, $\Psi \circ \Phi = \mathrm{id}_{V \otimes U}$. Therefore,

$$U \otimes V \cong V \otimes U.$$

■

13.1.2. Tensor Product of Linear Transformation

Motivation. Given two linear transformations $T : V \to V'$ and $S : W \to W'$, we want to construct the tensor product

$$T \otimes S : V \otimes W \to V' \otimes W'$$

Question: is $T \otimes S$ a linear transformation?

Answer: Yes. Universal property plays a role!