12.4. Wednesday for MAT3040

Reviewing. Bilinear map: $f: V\times W \to U$, e.g.,

$$\begin{aligned} f: &\quad \mathbb{R}^3\times\mathbb{R}^3 \\ \text{with} &\quad f(u,v)=u\times v \end{aligned}$$

Note that f is usually not a linear transformation, e.g.,

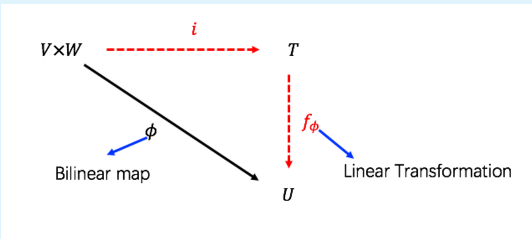
$$f(3(\boldsymbol{v},\boldsymbol{w})) = f(3\boldsymbol{v},3\boldsymbol{w}) = (3\boldsymbol{v})\times(3\boldsymbol{w}) = 9\boldsymbol{v}\times\boldsymbol{w} \neq 3f(\boldsymbol{v},\boldsymbol{w}).$$

The vector space structure of $\boldsymbol{v}\times\boldsymbol{w}$ is not suited to study bilinear map, and the proper way is to study its induced linear transformation.

Definition 12.3 [Universal Property of Tensor Product] Let V,W be vector spaces. Consider the set

$$\mathsf{Obj} := \{\phi: V\times W\to U \mid \phi \text{ is a bilinear map}\}$$

We say T, or $(i: V\times W\to T)\in\mathsf{Obj}$ satisfies the **universal property** if for any $(\phi: V\times W\to T)\in\mathsf{Obj}$, there exists an unique linear transformation $f_\phi: T\to U$ such that the diagram below commutes:



i.e., $\phi = f_\phi\circ i$.

Therefore, rather than studying bilinear map ϕ, it is better to study the linear transformation f_ϕ instead.

Question: does T exist?

Definition 12.4 [Spanning Set] Let V, W be vector spaces. Let $S = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \boldsymbol{v} \in V, \boldsymbol{w} \in W\}$, then we define

$$\mathfrak{X} = \text{span}(S).$$

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1. The spanning set $\mathfrak{X}$ is not addictive, e.g., $\mathfrak{x}_1 = 3(0, \boldsymbol{w}) \in \mathfrak{X}$ and $\mathfrak{x}_2 = 1(0, \boldsymbol{w}) + 1(0, 2\boldsymbol{w}) \in \mathfrak{X}$, but $\mathfrak{x}_1 \neq \mathfrak{x}_2$.
2. Note that we assume no relations on the elements $(\boldsymbol{v}, \boldsymbol{w}) \in \mathcal{S}$. More precisely, the set $S = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \boldsymbol{v} \in V, \boldsymbol{w} \in W\}$ is linearly independent in $\mathfrak{X}$. For example, $(0, \boldsymbol{w}) \perp (0, 2\boldsymbol{w})$.
3. The only legitimate relationship is

$$2(\boldsymbol{v}_1, \boldsymbol{w}_1) + 3(\boldsymbol{v}_1, \boldsymbol{w}_1) = 5(\boldsymbol{v}, \boldsymbol{w}),$$

   which is not equal to $(5\boldsymbol{v}, 5\boldsymbol{w})$
4. $\mathcal{S}$ is a basis of $\mathfrak{X}$, and therefore $\mathcal{X}$ is of uncountable dimension.

Definition 12.5 [Special subspace of $\mathfrak{X}$] Let $y \leq \mathfrak{X}$ be a vector subspace spanned by vectors of the form

$$\{1(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}) - 1(\boldsymbol{v}_1, \boldsymbol{w}) - 1(\boldsymbol{v}_2, \boldsymbol{w})\}, \quad \text{or} \quad \{1(\boldsymbol{v}, \boldsymbol{w}_1 + \boldsymbol{w}_2) - 1(\boldsymbol{v}, \boldsymbol{w}_1) - 1(\boldsymbol{v}, \boldsymbol{w}_2)\}$$

or

$$\{1(k\boldsymbol{v}, \boldsymbol{w}) - k(\boldsymbol{v}, \boldsymbol{w}) \mid k \in \mathbb{F}\}$$

or

$$\{1(\boldsymbol{v}, k\boldsymbol{w}) - k(\boldsymbol{v}, \boldsymbol{w}) \mid k \in \mathbb{F}\}$$

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Definition 12.6 [Tensor Product] We define the **tensor product** $V \otimes W$ by

$$V \otimes W = \mathcal{X}/y.$$

Therefore, $\boldsymbol{v} \otimes \boldsymbol{w} = (\boldsymbol{v}, \boldsymbol{w}) + y \in \mathcal{X}/y$

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1. As a result, the tensor product is finitely addictive:

$$\begin{aligned}(\boldsymbol{v}_1 + \boldsymbol{v}_2) \otimes \boldsymbol{w} &= (\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{w}) + y \\ &= (\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{w}) - [(\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{w}) - (\boldsymbol{v}_1, \boldsymbol{w}) - (\boldsymbol{v}_2, \boldsymbol{w})] + y \\ &= 0(\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{w}) + (\boldsymbol{v}_1, \boldsymbol{w}) + (\boldsymbol{v}_2, \boldsymbol{w}) + y \\ &= [(\boldsymbol{v}_1, \boldsymbol{w}) + y] + [(\boldsymbol{v}_2, \boldsymbol{w}) + y] \\ &= \boldsymbol{v}_1 \otimes \boldsymbol{w} + \boldsymbol{v}_2 \otimes \boldsymbol{w}\end{aligned}$$

   Similarly,

$$\begin{aligned}\boldsymbol{v} \otimes (\boldsymbol{w}_1 + \boldsymbol{w}_2) &= (\boldsymbol{v} \otimes \boldsymbol{w}_1) + (\boldsymbol{v} \otimes \boldsymbol{w}_2) \\ (k\boldsymbol{v}) \otimes \boldsymbol{w} &= k(\boldsymbol{v} \otimes \boldsymbol{w}) \\ \boldsymbol{v} \otimes (k\boldsymbol{w}) &= k(\boldsymbol{v} \otimes \boldsymbol{w})\end{aligned}$$

2. The product space $V \times W$ is different from the tensor product space $V \otimes W$:

   (a) $(\boldsymbol{v}, \boldsymbol{0}) \neq \boldsymbol{0}_{V \times W}$ in $V \times W$; but $\boldsymbol{v} \otimes 0 \in 0_{V \otimes W}$:

$$\begin{aligned}V \otimes 0 &= V \otimes (0\boldsymbol{w}) \\ &= 0(V \otimes w) \\ &= 0_{V \otimes W}\end{aligned}$$

   Moreover, f is bilinear implies $f(\boldsymbol{v}, 0) = \boldsymbol{0}$.

(b) $(\boldsymbol{v}_1, \boldsymbol{w}_1) + (\boldsymbol{v}_2, \boldsymbol{w}_2) = (\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{w}_1 + \boldsymbol{w}_2)$; but $\boldsymbol{v}_1 \otimes \boldsymbol{w}_1 + \boldsymbol{v}_2 \otimes \boldsymbol{w}_2$ cannot be simplified further, unless $\boldsymbol{v}_1 = \boldsymbol{v}_2$:

$$\boldsymbol{v} \otimes \boldsymbol{w}_1 + \boldsymbol{v} \otimes \boldsymbol{w}_2 = \boldsymbol{v} \otimes (\boldsymbol{w}_1 + \boldsymbol{w}_2)$$

Theorem 12.3 The bilinear map

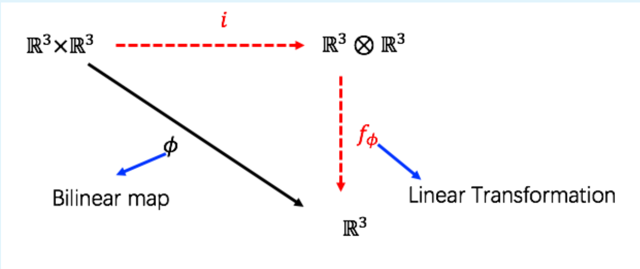
$$\begin{aligned} i: \quad & V \times W \to V \otimes W \quad (i \in \mathrm{Obj}) \\ \text{with} \quad & (\boldsymbol{v}, \boldsymbol{w}) \mapsto \boldsymbol{v} \otimes \boldsymbol{w} \end{aligned}$$

satisfies the universal property of tensor products.

■ **Example 12.6** Consider a common bilinear map

$$\begin{aligned} \phi: \quad & \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \\ \text{with} \quad & (\boldsymbol{v}, \boldsymbol{w}) \mapsto \boldsymbol{v} \times \boldsymbol{w} \end{aligned}$$

By the universal property, there exists the linear transformation $f_\phi : \mathbb{R}^3 \otimes \mathbb{R}^3 \to \mathbb{R}^3$ such that the diagram below commutes:



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