Chapter 12

Week12

12.1. Monday for MAT3040

12.1.1. Remarks on Normal Operator

Proposition 12.1 If T is normal, then

1. $\|T(\boldsymbol{v})\| = \|T'(\boldsymbol{v})\|$ for any $\boldsymbol{v} \in V$
2. $(T - \lambda I)$ is normal for any $\lambda \in \mathbb{C}$
3. $T(\boldsymbol{v}) = \lambda \boldsymbol{v}$ if and only if $T'(\boldsymbol{v}) = \bar{\lambda}\boldsymbol{v}$
4. If $T(\boldsymbol{v}) = \lambda \boldsymbol{v}$ and $T(\boldsymbol{w}) = \mu \boldsymbol{w}$ with $\lambda \neq \mu$, then $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$.

Proof. (3)
- For the forward direction, if $(T - \lambda I)\boldsymbol{v} = 0$, then by part (2), $(T - \lambda I)$ is normal, which follows that

$$\|(T - \lambda I)'(\boldsymbol{v})\| = 0 \implies (T - \lambda I)'(\boldsymbol{v}) = 0 \implies T'\boldsymbol{v} = \bar{\lambda}\boldsymbol{v}.$$

- For the reverse direction, suppose that $(T' - \bar{\lambda} I)\boldsymbol{v} = 0$. Since T is normal, we imply T' is normal. Then by part (2), $(T' - \bar{\lambda} I)$ is normal. By applying the same trick,

$$(T' - \bar{\lambda} I)'\boldsymbol{v} = 0 \implies ((T')' - \bar{\bar{\lambda}} I)\boldsymbol{v} = 0.$$

By hw4, $(T')' = T$. Therefore, $(T - \lambda I)\boldsymbol{v} = 0$.

(4) Observe that

$$\lambda\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \bar{\lambda}\boldsymbol{v}, \boldsymbol{w} \rangle \xRightarrow{\text{by (3)}} \lambda\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle T'(\boldsymbol{v}), \boldsymbol{w} \rangle = \langle \boldsymbol{v}, T(\boldsymbol{w}) \rangle = \langle \boldsymbol{v}, \mu \boldsymbol{w} \rangle = \mu \langle \boldsymbol{v}, \boldsymbol{w} \rangle$$

Since $\lambda \neq \mu$, we imply $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$. The proof is complete.

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Theorem 12.1 Let T be an operator on a finite dimensional ($\dim(V) = n$) $\mathbb{C}$-inner product vector space V satisfying $T'T = TT'$. Then there is an orthonormal basis of eigenvectors of V, i.e., an orthonormal basis of V such that any element from this basis is an eigenvector of T.

Proof. Since $\mathcal{X}_T(x)$ must have a root in $\mathbb{C}$, there must exist an eigen-pair $(\boldsymbol{v}, \lambda)$ of T.

- Construct $U = \text{span}\{\boldsymbol{v}\}$, and it follows that

$$T\boldsymbol{v} = \lambda\boldsymbol{v} \implies U \text{ is } T\text{-invariant.}$$

$$T'\boldsymbol{v} = \bar{\lambda}\boldsymbol{v} \implies U \text{ is } T'\text{-invariant.}$$

- Moreover, we claim that $U^\perp$ is T and T' invariant: let $\boldsymbol{w} \in U^\perp$, and for all $\boldsymbol{u} \in U$, we have

$$\langle \boldsymbol{u}, T(\boldsymbol{w}) \rangle = \langle T'(\boldsymbol{u}), \boldsymbol{w} \rangle = \langle \bar{\lambda}\boldsymbol{u}, \boldsymbol{w} \rangle = \lambda \langle \boldsymbol{u}, \boldsymbol{w} \rangle = 0,$$

i.e., $U^\perp$ is T invariant.

$$\langle \boldsymbol{u}, T'(\boldsymbol{w}) \rangle = \langle T(\boldsymbol{u}), \boldsymbol{w} \rangle = \langle \lambda\boldsymbol{u}, \boldsymbol{w} \rangle = \bar{\lambda} \langle \boldsymbol{u}, \boldsymbol{w} \rangle = 0,$$

which implies $U^\perp$ is T' invariant.

- Therefore, we construct the operator $T|_{U^\perp} : U^\perp \to U^\perp$, and

$$TT' = T'T \implies (T|_{U^\perp})(T'|_{U^\perp}) = (T'|_{U^\perp})(T|_{U^\perp}),$$

i.e., $(T|_{U^\perp})$ is normal on $U^\perp$. Moreover, $\dim(U^\perp) = n - 1$.

- Applying the same trick as in Theorem (**??**), we imply there exists an orthonormal basis $\{\boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ of eigenvectors of $(T\mid_{U^\perp})$. Then we can argue that

$$\mathcal{B} = \{\boldsymbol{v}' = \boldsymbol{v}/\|\boldsymbol{v}\|, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{k+1}\}$$

is a basis of orthonormal eigenvectors of V.

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Corollary 12.1 [Spectral Theorem for Normal Operator] Let $T : V \to V$ be a normal operator on a $\mathbb{C}$-inner product space with $\dim(V) < \infty$. Then there exists self-adjoint operators $P_1, \ldots, P_k$ such that

$$P_i^2 = P_i, \quad P_i P_j = 0, i \neq j, \quad \sum_{i=1}^{k} P_i = I,$$

and $T = \sum_{i=1}^{k} \lambda_i P_i$, where λ_i's are the eigenvalues of T.

R These P_i's are the **orthogonal projections** from V to the λ_i-eigenspace $\ker(T - \lambda_i I)$ of T, denoted as

$$P_i = \prod_{\ker(T-\lambda_i I)} (T), \; i = 1, \ldots, k.$$

You should know how to compute P_i's when $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$ in the course MAT2040.

Proof. Since T has a basis of eigenvectors, by definition, T is diagonalizable. By proposition (8.2),

$$m_T(x) = (x - \lambda_1) \cdots (x - \lambda_k),$$

where λ_i's are distinct. By spectral decomposition corollary (**??**), it suffices to show P_i's are self-disjoint.

- Recall that $P_i = a_i(T)q_i(T) := b_m T^m + \cdots + b_1 T + b_0 T$, i.e., a polynomial of T, and therefore

$$P_i' = \bar{b}_m (T')^m + \cdots + \bar{b}_1 (T') + \bar{b}_0 I.$$

We claim that P_i is normal: Since $T'T = TT'$, we imply

$$(T')^p T^q = T^q (T')^p, \forall p,q \in \mathbb{N}$$

which follows that

$$\begin{aligned}
P_i P_i' &= (b_m T^m + \cdots + b_0 I)(\bar{b}_m (T')^m + \cdots + \bar{b}_1 (T') + \bar{b}_0 I) \\
&= \sum_{1 \le x,y \le m} b_x \bar{b}_y (T)^x (T')^y \\
&= \sum_{1 \le x,y \le m} \bar{b}_y b_x (T')^y (T)^x \\
&= (\bar{b}_m (T')^m + \cdots + \bar{b}_1 (T') + \bar{b}_0 I)(b_m T^m + \cdots + b_0 I) \\
&= P_i' P_i
\end{aligned}$$

- In general, S is self-adjoint, which implies S is normal, but not vice versa. However, the converse holds if further all eigenvalues of S are real numbers:

  By Theorem (12.1), we imply S is orthonormally diagonalizable, and its diagonal representation is of the form

  $$(S)_{\mathcal{B},\mathcal{B}} = \text{diag}(\lambda_1, \dots, \lambda_k).$$

  Note that $\mathcal{B}$ is also a basis for S' and elements of $\mathcal{B}$ are eigenvalues of S', by part (3) in proposition (12.1). Therefore,

  $$(S')_{\mathcal{B},\mathcal{B}} = \text{diag}(\lambda_1, \dots, \lambda_k).$$

  Therefore, $S = S'$.

In particular, for $S = P_i$, we can easily show all eigenvalues of P_i are 0 or 1, which are real. Therefore, P_i's are self-adjoint. ∎

Corollary 12.2 Let $T: V \to V$ be a linear operator on $\mathbb{C}$-inner product space with $\dim(V) < \infty$. Then T is normal if and only if $T' = f(T)$ for some polynomial $f(x) \in \mathbb{C}[x]$.

Proof.
- For the reverse direction, if $T' = f(T)$, then $T'T = f(T)T = Tf(T) = TT'$.
- For the forward direction, suppose that T is normal, then by corollary (12.1),

$$T = \sum_{i=1}^{k} \lambda_i P_i, \ P_i = f_i(T), \text{ where } P_i\text{'s are self-adjoint,}$$

which follows that

$$T' = \left(\sum_{i=1}^{k} \lambda_i P_i\right)' = \sum_{i=1}^{k} \bar{\lambda}_i P_i' = \sum_{i=1}^{k} \bar{\lambda}_i P_i = \sum_{i=1}^{k} \bar{\lambda}_i f_i(T)$$

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(R) The normal operator is a generalization of Hermitian matrices, and it inherits many nice properties of Hermitian.

12.1.2. Tensor Product

Motivation. Let U, V, W be vector spaces. We want to study bilinear maps $f: U \times W \to U$, i.e.,

$$f(av_1 + bv_2, w) = af(v_1, w) + bf(v_2, w)$$

$$f(v, cw_1 + dw_2) = cf(v, w_1) + df(v, w_2)$$

Unfortunately, bilinear form usually is not a linear transformation!

▪ **Example 12.1**
- Let $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be with $(u, v) \mapsto \langle u, v \rangle$.
- Let $f: M_{n\times n}(\mathbb{F}) \times M_{n\times n}(\mathbb{F}) \to M_{n\times n}(\mathbb{F})$ be with $f(A, B) = AB$.
- Let $f: \mathbb{F}[x] \times \mathbb{F}[x] \to \mathbb{F}$ be with $f(p(x), q(x)) = p(1)q(2)$

- Let $f : \mathbb{F}[x] \times \mathbb{F}[x] \to \mathbb{F}[x]$ be with $f(p(x), q(x)) = p(x)q(x)$.