Chapter 11

Week11

11.1. Monday for MAT3040

Reviewing. Adjoint Operator: $\langle T'(\boldsymbol{v}),\boldsymbol{w}\rangle = \langle \boldsymbol{v},T(\boldsymbol{w})\rangle$.

11.1.1. Self-Adjoint Operator

Definition 11.1 [Self-Adjoint] Let V be an inner product space and $T: V \to V$ be a linear operator. Then T is **self-adjoint** if $T' = T$.

Example 11.1 Let $V = \mathbb{C}^n$, and $\mathcal{B} = \{\boldsymbol{e}_1,\dots,\boldsymbol{e}_n\}$ be a orthonormal basis. Let $T: V \to V$ be given by

$$T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}, \quad \text{where } A \in M_{n\times n}(\mathbb{C}).$$

Or equivalently, there exists basis $\mathcal{B}$ such that $(T)_{\mathcal{B},\mathcal{B}} = \boldsymbol{A}$.

In such case, T is self-adjoint if and only if $(T')_{\mathcal{B},\mathcal{B}} = (T)_{\mathcal{B},\mathcal{B}}$, i.e., $\overline{(T)^{\mathrm{T}}_{\mathcal{B},\mathcal{B}}} = (T)_{\mathcal{B},\mathcal{B}}$, i.e., $\boldsymbol{A}^{\mathrm{H}} = \boldsymbol{A}$.

Therefore, $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$ is self-adjoint if and only if $\boldsymbol{A}^{\mathrm{H}} = \boldsymbol{A}$.

Moreover, if $\mathbb{C}$ is replaced by $\mathbb{R}$, then T is seld-adjoint if and only if $\boldsymbol{A}$ is symmetric.

R The notion of self-adjoint for linear operator is essentially the generalized notion of Hermitian for matrix that we have stuided in MAT2040.

We also have some nice properties for self-adjoint, and the proof for which are essentially the same for the proof in the case of Hermitian matrices.

Proposition 11.1 If λ is an eigenvalue of a self-adjoint operator T, then $\lambda \in \mathbb{R}$.

Proof. Suppose there is an eigen-pair $(\lambda, \boldsymbol{w})$ for $\boldsymbol{w} \neq \boldsymbol{0}$, then

$$\begin{aligned}\lambda\langle \boldsymbol{w}, \boldsymbol{w}\rangle &= \langle \boldsymbol{w}, \lambda \boldsymbol{w}\rangle \\ &= \langle \boldsymbol{w}, T(\boldsymbol{w})\rangle = \langle T'(\boldsymbol{w}), \boldsymbol{w}\rangle \\ &= \langle T(\boldsymbol{w}), \boldsymbol{w}\rangle = \langle \lambda \boldsymbol{w}, \boldsymbol{w}\rangle \\ &= \bar{\lambda}\langle \boldsymbol{w}, \boldsymbol{w}\rangle\end{aligned}$$

Since $\langle \boldsymbol{w}, \boldsymbol{w}\rangle \neq 0$ by non-degeneracy property, we have $\lambda = \bar{\lambda}$, i.e., $\lambda \in \mathbb{R}$. ■

Proposition 11.2 If $U \leq V$ is T-invariant over the self-adjoint operator T, then so is $U^{\perp}$.

Proof. It suffices to show $T(\boldsymbol{v}) \in U^{\perp}, \forall \boldsymbol{v} \in U^{\perp}$, i.e., for any $\boldsymbol{u} \in U$, check that

$$\langle \boldsymbol{u}, T(\boldsymbol{v})\rangle = \langle T'(\boldsymbol{u}), \boldsymbol{v}\rangle = \langle T(\boldsymbol{u}), \boldsymbol{v}\rangle = 0,$$

where the last equality is because that $T(\boldsymbol{u}) \in U$ and $\boldsymbol{v} \in U^{\perp}$. Therefore, $T(\boldsymbol{v}) \in U^{\perp}$. ■

Theorem 11.1 If $T : V \to V$ is self-adjoint, and $\dim(V) < \infty$, then there exists an orthonormal basis of eigenvectors of T, i.e., an orthonormal basis of V such that any element from this basis is an eigenvector of T.

Proof. We use the induction on $\dim(V)$:

- The result is trival for $\dim(V) = 1$.
- Suppose that this theorem holds for all vector spaces V with $\dim(V) \leq k$, then we want to show the theorem holds when $\dim(V) = k+1$:

  Suppose that $T : V \to V$ is self-adjoint with $\dim(V) = k+1$, then consider

$$\mathcal{X}_T(x) = x^{k+1} + \cdots + a_1 x + a_0, \quad a_i \in \mathbb{K}, \text{ where } \mathbb{K} \text{ denotes } \mathbb{R} \text{ or } \mathbb{C}.$$

- If $\mathbb{K} = \mathbb{C}$, then $\mathcal{X}_T(x)$ can be decomposed as

$$\mathcal{X}_T(x) = (x - \lambda_1)\cdots(x - \lambda_{k+1})$$

  In paricular, we obtain the eigen-pair $(\lambda_1, \boldsymbol{v})$

- If $\mathbb{K} = \mathbb{R}$, i.e., we treat real number as scalars, then

$$\mathcal{X}_T(x) = (x - \lambda_1)\cdots(x - \lambda_{k+1}), \text{ where } \lambda_i \in \mathbb{C}.$$

  However, we should argue that $\lambda_1 \in \mathbb{R}$, since otherwise λ_1 cannot be treated as scalars, which makes the following arguments invalid.

  By proposition (11.1), we imply all λ_i's are in $\mathbb{R}$. Moreover, we also obtain the eigen-pair $(\lambda_1, \boldsymbol{v})$

Consider $U = \text{span}\{\boldsymbol{v}\}$, then

- U is T-invariant
- $V = U \oplus U^{\perp}$, since V is finite dimensional
- $U^{\perp}$ is T-invariant.

Consider $T\mid_{U^{\perp}}$, which is a self-adjoint operator on $U^{\perp}$, with $\dim(U^{\perp}) = k + 1$.

By induction, there exists an orthonormal basis $\{\boldsymbol{e}_2, \dots, \boldsymbol{e}_{k+1}\}$ of eigenvectors of $T\mid_{U^{\perp}}$.

Consider the basis $\mathcal{B} = \{\boldsymbol{v}' = \boldsymbol{v}/\|\boldsymbol{v}\|, \boldsymbol{e}_2, \dots, \boldsymbol{e}_{k+1}\}$. As a result,

1. $\mathcal{B}$ forms a basis of V
2. All $\boldsymbol{v}', \boldsymbol{e}_i$ are of norm 1 eigenvectors of T.
3. $\mathcal{B}$ is an orthonormal set, e.g., $\langle \boldsymbol{v}', \boldsymbol{e}_i \rangle = 0$, where $\boldsymbol{v}' \in U$ and $\boldsymbol{e}_i \in U^{\perp}$.

Therefore, $\mathcal{B}$ is a basis of orthonormal eigenvectors of V.

$\blacksquare$

Corollary 11.1 If $\dim(V) < \infty$, and $T : V \to V$ is self-adjoint, then there exists orthonormal basis $\mathcal{B}$ such that

$$(T)_{\mathcal{B},\mathcal{B}} = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

In paticular, for all real symmtric matrix $\boldsymbol{A} \in \mathbb{S}^n$, there exists orthogonal matrix P ($P^{\mathrm{T}}P = \boldsymbol{I}_n$) such that

$$P^{-1}AP = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

Proof. 1. By applying theorem (11.1), there exists orthonormal basis of V, say $\mathcal{B} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ such that $T(\boldsymbol{v}_i) = \lambda_i \boldsymbol{v}_i$. Directly writing the basis representation gives

$$(T)_{\mathcal{B},\mathcal{B}} = \mathrm{diag}(\lambda_1, \dots, \lambda_n).$$

2. For the second part, consider $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$. Since $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$, we imply T is self-adjoint. There exists orthonormal basis $\mathcal{B} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ such that

$$(T)_{\mathcal{B},\mathcal{B}} = \mathrm{diag}(\lambda_1, \dots, \lambda_n).$$

In particular, if $\mathcal{A} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$, then $(T)_{\mathcal{A},\mathcal{A}} = \boldsymbol{A}$. We construct $P := C_{\mathcal{A},\mathcal{B}}$, which is the change of basis matrix from $\mathcal{B}$ to $\mathcal{A}$, then

$$P = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix}$$

and

$$P^{-1}(T)_{\mathcal{A},\mathcal{A}}P = (T)_{\mathcal{B},\mathcal{B}}$$

Or equivalently, $P^{-1}AP = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, with

$$P^{\mathrm{T}}P = \begin{pmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \vdots \\ \boldsymbol{v}_n^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix} = \boldsymbol{I}$$

■

11.1.2. Orthononal/Unitary Operators

Definition 11.2 A linear operator $T : V \to V$ over $\mathbb{K}$ with $\langle T(\boldsymbol{w}), T(\boldsymbol{v})\rangle = \langle \boldsymbol{w}, \boldsymbol{v}\rangle, \forall \boldsymbol{v}, \boldsymbol{w} \in V$, is called

1. **Orthogonal** if $\mathbb{K} = \mathbb{R}$
2. **Unitary** if $\mathbb{K} = \mathbb{C}$

Proposition 11.3 T is orthogonal / unitary if and only if $T' \circ T = I$

Proof. The reverse direction is by directly checking that

$$\langle T(\boldsymbol{w}), T(\boldsymbol{v})\rangle = \langle T' \circ T(\boldsymbol{w}), \boldsymbol{v}\rangle = \langle \boldsymbol{w}, \boldsymbol{v}\rangle$$

The forward direction is by checking $T' \circ T(\boldsymbol{w}) = \boldsymbol{w}, \forall \boldsymbol{w} \in V$:

$$\langle T' \circ T(\boldsymbol{w}), \boldsymbol{v}\rangle = \langle T(\boldsymbol{w}), T(\boldsymbol{v})\rangle = \langle \boldsymbol{w}, \boldsymbol{v}\rangle \implies \langle T' \circ T(\boldsymbol{w}) - \boldsymbol{w}, \boldsymbol{v}\rangle = 0, \forall \boldsymbol{v} \in V$$

By non-degeneracy, $T' \circ T(\boldsymbol{w}) - \boldsymbol{w} = 0$, i.e., $T' \circ T(\boldsymbol{w}) = \boldsymbol{w}, \forall \boldsymbol{w} \in V$. ■

Example 11.2 Let $T : \mathbb{K}^n \to \mathbb{K}^n$ be given by $T(\boldsymbol{v}) = A\boldsymbol{v}$. Then T is orthogonal implies $(T')_{\mathcal{B},\mathcal{B}}(T)_{\mathcal{B},\mathcal{B}} = I$.

(Orthogonal) When $\mathbb{K} = \mathbb{R}$, then $A^{\mathrm{T}}A = I$

(Unitary) When $\mathbb{K} = \mathbb{C}$, then $A^{\mathrm{H}}A = I$.

Definition 11.3 [Orthogonal/Unitary Group]

$$\text{Orthognoal Group} : O(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid A^{\mathrm{T}}A = I\}$$

$$\text{Unitary Group} : O(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{C}) \mid A^{\mathrm{H}}A = I\}$$