10.2. Wednesday for MAT3040

Reviewing. Consider the mapping

$$\begin{aligned} \phi: &\quad V \to V^* \\ \text{with} &\quad \phi(\boldsymbol{v}) = \phi_{\boldsymbol{v}} \\ \text{where} &\quad \phi_{\boldsymbol{v}}(\boldsymbol{w}) = \langle \boldsymbol{v}, \boldsymbol{w} \rangle \end{aligned}$$

The Riesz Representation Theorem claims that

1. ϕ is a $\mathbb{R}$-linear transformation.
2. ϕ is injective.
3. If $\dim(V) < \infty$, then ϕ is an isomorphism.

Proof for Claim (2). Consider the equality $\phi(\boldsymbol{v}) = \phi_{\boldsymbol{v}} = 0_{V^*}$, which implies

$$\phi_{\boldsymbol{v}}(\boldsymbol{w}) = \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0, \forall \boldsymbol{w} \in V$$

By the non-degenercy property, $\boldsymbol{v} = 0_{\boldsymbol{v}}$, i.e., ϕ is injective. ■

Proof for Claim (3). Since $\dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(V^*)$, and ϕ is injective as a $\mathbb{R}$-linear transformation, we imply ϕ is an isomorphism from V to V^*, where V, V^* are treated as vector spaces over $\mathbb{R}$. ■

10.2.1. Orthogonal Complement

Definition 10.5 [Orthogonal Complement] Let $U \le V$ be a subspace of an inner product space. Then the **orthogonal complement** of U is

$$U^{\perp} = \{\boldsymbol{v} \in V \mid \langle \boldsymbol{v}, \boldsymbol{u} \rangle = 0, \forall \boldsymbol{u} \in U\}$$

The analysis for orthogonal complement for vector spaces over $\mathbb{C}$ is quite similar as what we have studied in MAT2040.

Proposition 10.1 1. $U^\perp$ is a subspace of V

2. $U \cap U^\perp = \{0\}$
3. $U_1 \subseteq U_2$ implies $U_2^\perp \leq U_1^\perp$.

Proof. 1. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2 \in U^\perp$, where $a, b \in K$ ($K = \mathbb{C}$ or $\mathbb{R}$), then for all $\boldsymbol{u} \in U$,

$$\begin{aligned}\langle a\boldsymbol{v}_1 + b\boldsymbol{v}_2, \boldsymbol{u}\rangle &= \bar{a}\langle \boldsymbol{v}_1, \boldsymbol{u}\rangle + \bar{b}\langle \boldsymbol{v}_2, \boldsymbol{u}\rangle \\ &= \bar{a} \cdot 0 + \bar{b} \cdot 0 = 0\end{aligned}$$

Therefore, $a\boldsymbol{v}_1 + b\boldsymbol{v}_2 \in U^\perp$.

2. Suppose that $\boldsymbol{u} \in U \cap U^\perp$, then we imply $\langle \boldsymbol{u}, \boldsymbol{u}\rangle = 0$. By the positive-definiteness of inner product, $\boldsymbol{u} = \mathbf{0}$.
3. The statement (3) is easy.

■

Proposition 10.2 1. If $\dim(V) < \infty$ and $U \leq V$, then $V = U \oplus U^\perp$

2. If $U, W \leq V$, then

$$\begin{aligned}(U + W)^\perp &= U^\perp \cap W^\perp \\ (U \cap W)^\perp &\supseteq U^\perp + W^\perp \\ (U^\perp)^\perp &\supseteq U\end{aligned}$$

Moreover, if $\dim(V) < \infty$, then these are equalities.

Proof. 1. Suppose that $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ forms a basis for U, and by basis extension, we obtain $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_n\}$ is a basis for V.

By Gram-Schmidt Process, any finite basis induces an orthonormal basis.

Therefore, suppose that $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_k\}$ forms an orthonormal basis for U, and $\{\boldsymbol{e}_{k+1}, \dots, \boldsymbol{e}_n\}$ forms an orthonormal basis for $U^\perp$.

It's easy to show $V = U + U^\perp$ using orthonormal basis.

2. (a) The reverse part $(U + W)^\perp \supseteq U^\perp \cap W^\perp$ is trivial; for the forward part, suppose

$\boldsymbol{v} \in (U+W)^{\perp}$, then

$$\langle \boldsymbol{v}, \boldsymbol{u}+\boldsymbol{w}\rangle = 0, \ \forall \boldsymbol{u} \in U, \ \boldsymbol{w} \in W$$

Taking $\boldsymbol{u} \equiv \boldsymbol{0}$ in the equality above gives $\langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0$, i.e., $\boldsymbol{v} \in U^{\perp}$. Similarly, $\boldsymbol{v} \in W^{\perp}$.

(b) Follow the similar argument as in (2a). If $\dim(V) < \infty$, then write down the orthonormal basis for $U^{\perp} + W^{\perp}$ and $(U \cap W)^{\perp}$.

(c) Follow the similar argument as in (2a). If $\dim(V) < \infty$, then

$$V = U^{\perp} \oplus (U^{\perp})^{\perp} = U \oplus U^{\perp}.$$

Therefore, $(U^{\perp})^{\perp} = U$.

■

Proposition 10.3 The mapping $\phi : V \to V^*$ maps $U^{\perp} \le V$ injectively to $\text{Ann}(U) \le V^*$. If $\dim(V) < \infty$, then $U^{\perp} \cong \text{Ann}(U)$ as $\mathbb{R}$-vector spaces

Proof. The injectivity of ϕ has been shown at the beginning of this lecture. For any $\boldsymbol{v} \in U^{\perp}$, we imply $\phi_{\boldsymbol{v}}(\boldsymbol{u}) = 0, \forall \boldsymbol{u} \in U$, i.e., $\phi_{\boldsymbol{v}} \in \text{Ann}(U)$.

Therefore, $\phi(U^{\perp}) \le \text{Ann}(U)$.

Provided that $\dim(V) < \infty$, by (1) in proposition (10.2),

$$\dim(U) + \dim(U^{\perp}) = \dim(V)$$

Since $\dim(U) + \dim(\text{Ann}(U)) = \dim(V)$, we imply $\dim(U^{\perp}) = \dim(\text{Ann}(U))$.

Moreover,

$$\phi : U^{\perp} \to \text{Ann}(U)$$

is an isomorphism between $\mathbb{R}$-vector spaces $U^{\perp}$ and $\text{Ann}(U)$. ■

10.2.2. Adjoint Map

Motivation. Then we study the induced mapping based on a given linear operator T, denoted as T'. This induced mapping essentially plays the similar role as taking the Hermitian for a complex matrix.

Notation. Previously we have studied the **adjoint** of $T : V \to W$, denoted as $T^* : W^* \to V^*$. However, from now on, we use the same terminology but with different meaning. If $T : V \to V$ is a linear operator, then the **adjoint** of T is the linear operator $T^* : W^* \to V^*$ defined as follows.

Definition 10.6 [Adjoint] Let $T : V \to V$ be a linear operator between inner product spaces. The **adjoint** of T is defined as $T' : V \to V$ satisfying

$$\langle T'(\boldsymbol{v}), \boldsymbol{w}\rangle = \langle \boldsymbol{v}, T(\boldsymbol{w})\rangle, \ \forall \boldsymbol{w} \in V \tag{10.1}$$

Proposition 10.4 If $\dim(V) < \infty$, then T' exists, and it is unique. Moreove, T' is a linear map.

Proof. Fix any $\boldsymbol{v} \in V$. Consider the mapping

$$\alpha_{\boldsymbol{v}} : \boldsymbol{w} \xrightarrow{T} T(\boldsymbol{w}) \xrightarrow{\phi_{\boldsymbol{v}}} \langle \boldsymbol{v}, T(\boldsymbol{w})\rangle$$

This is a linear transformation from V to $\mathbb{F}$, i.e., $\alpha_{\boldsymbol{v}} \in V^*$

By Riesz representation theorem, ϕ is an isomorphism from V to V^*. Moreover, $\phi(T'(\boldsymbol{v})) = \alpha_{\boldsymbol{v}}$. Therefore, for any $\alpha_{\boldsymbol{v}} \in V^*$, there exists a vector $T'(\boldsymbol{v}) \in V$ such that

$$\phi(T'(\boldsymbol{v})) = \alpha_{\boldsymbol{v}} \in V^*$$

Or equivalently, $\phi_{T'(\boldsymbol{v})}(\boldsymbol{w}) = \alpha_{\boldsymbol{v}}(\boldsymbol{w}), \forall \boldsymbol{w} \in V$, i.e., $\langle T'(\boldsymbol{v}), \boldsymbol{w}\rangle = \langle \boldsymbol{v}, T(\boldsymbol{w})\rangle$.

Henceforce, from $\boldsymbol{v}$ we have constructed $T'(\boldsymbol{v})$ satisfying (10.1). Now define $T' : V \to V$ by $\boldsymbol{v} \mapsto T'(\boldsymbol{v})$.

- Since the choice of $T'(\boldsymbol{v})$ is unique by the injectivity of ϕ, T' is well-defined.
- Now we show T' is a linear transformation: Let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V, a, b \in K$. For all $\boldsymbol{w} \in V$, we have

$$\begin{aligned}\langle T'(a\boldsymbol{v}_1 + b\boldsymbol{v}_2), \boldsymbol{w}\rangle &= \langle a\boldsymbol{v}_1 + b\boldsymbol{v}_2, T(\boldsymbol{w})\rangle \\ &= \bar{a}\langle \boldsymbol{v}_1, T(\boldsymbol{w})\rangle + \bar{b}\langle \boldsymbol{v}_2, T(\boldsymbol{w})\rangle \\ &= \bar{a}\langle T'(\boldsymbol{v}_1), \boldsymbol{w}\rangle + \bar{b}\langle T'(\boldsymbol{v}_2), \boldsymbol{w}\rangle \\ &= \langle aT'(\boldsymbol{v}_1) + bT'(\boldsymbol{v}_2), \boldsymbol{w}\rangle\end{aligned}$$

Therfore,

$$\langle T'(a\boldsymbol{v}_1 + b\boldsymbol{v}_2) - [aT'(\boldsymbol{v}_1) + bT'(\boldsymbol{v}_2)], \boldsymbol{w}\rangle = 0, \ \forall \boldsymbol{w} \in V$$

By the non-degeneracy of inner product,

$$T'(a\boldsymbol{v}_1 + b\boldsymbol{v}_2) - [aT'(\boldsymbol{v}_1) + bT'(\boldsymbol{v}_2)] = \boldsymbol{0},$$

i.e., $T'(a\boldsymbol{v}_1 + b\boldsymbol{v}_2) = aT'(\boldsymbol{v}_1) + bT'(\boldsymbol{v}_2)$

∎

▪ **Example 10.2** Let $V = \mathbb{R}^n$, $\langle\cdot,\cdot\rangle$ as the usual inner product. Consider the matrix-multiplication mapping

$$\begin{aligned}T: \quad & V \to V \\ & T(\boldsymbol{v}) = A\boldsymbol{v}\end{aligned}$$

Then $\langle T'(\boldsymbol{v}), \boldsymbol{w}\rangle = \langle \boldsymbol{v}, T(\boldsymbol{w})\rangle$ implies

$$\begin{aligned}(T'(\boldsymbol{v}))^{\mathrm{T}}\boldsymbol{w} &= \langle \boldsymbol{v}, \boldsymbol{A}\boldsymbol{w}\rangle \\ &= \boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{w} \\ &= (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{v})^{\mathrm{T}}\boldsymbol{w}\end{aligned}$$

Therfore, $T'(\boldsymbol{v}) = A^{\mathrm{T}}\boldsymbol{v}$. ▪

Proposition 10.5 Let $T : V \to V$ be a linear transformation, V a inner product space. Suppose that $\mathcal{B} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is an orthonormal basis of V, then

$$(T')_{\mathcal{B},\mathcal{B}} = \overline{((T)_{\mathcal{B},\mathcal{B}})^{\mathrm{T}}}$$

Proof. Suppose that $(T)_{\mathcal{B},\mathcal{B}} = (a_{ij})$, where $T(\boldsymbol{e}_j) = \sum_{k=1}^{n} a_{kj}\boldsymbol{e}_k$, then

$$\begin{aligned}\langle \boldsymbol{e}_i, T(\boldsymbol{e}_j)\rangle &= \langle \boldsymbol{e}_i, \sum_{k=1}^{n} a_{kj}\boldsymbol{e}_k\rangle \\ &= \sum_{k=1}^{n} a_{kj}\langle \boldsymbol{e}_i, \boldsymbol{e}_k\rangle \\ &= a_{ij}\end{aligned}$$

Also, suppose $(T')_{\mathcal{B},\mathcal{B}} = (b_{ij})$, we imply $T'(\boldsymbol{e}_j) = \sum_{k=1}^{n} b_{ij}\boldsymbol{e}_k$, which follows that

$$\langle \boldsymbol{e}_i, T'(\boldsymbol{e}_j)\rangle = b_{ij} \implies \overline{\langle T'(\boldsymbol{e}_j), \boldsymbol{e}_i\rangle} = b_{ij} \implies \overline{\langle \boldsymbol{e}_j, T(\boldsymbol{e}_i)\rangle} = b_{ij},$$

i.e., $\overline{a_{ji}} = b_{ij}$. ∎

R Proposition (10.5) does not hold if $\mathcal{B}$ is not an orthonormal basis.