9.5. Wednesday for MAT3006

9.5.1. Remarks on Measurable function

Proposition 9.6 Let f_n be a sequence of measurable functions $f_n : \mathbb{R} \to [-\infty, \infty]$. Then the functions

$$\sup_{n\in\mathbb{N}} f_n(x), \quad \inf_{n\in\mathbb{N}} f_n(x), \quad \lim_{n\to\infty} \sup f_n(x), \quad \lim_{n\to\infty} \inf f_n(x)$$

are measurable.

Proof.
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$$\begin{aligned}\left(\sup_{n\in\mathbb{N}} f_n\right)^{-1}((a,\infty]) &= \{x \in \mathbb{R} \mid \sup_n f_n(x) > a\} \\ &= \{x \in \mathbb{R} \mid f_n(x) > a \text{ for some } a\} \\ &= \bigcup_{n\in\mathbb{N}} f_n^{-1}((a,\infty])\end{aligned}$$

  which is measurable due to the measurability of f_n.

- The proof for the measurablitiy of $\inf_n f_n(x), \lim_{n\to\infty} \sup f_n(x), \lim_{n\to\infty} \inf f_n(x)$ is directly by applying the formula

$$\begin{aligned}\inf f_n(x) &= -(\sup(-f_n(x))) \\ \lim_{n\to\infty} \sup f_n(x) &= \lim_{m\to\infty} (\sup_{n\geq m} f_n(x)) = \inf_{m\in\mathbb{N}} (\sup_{n\geq m} f_n(x)) \\ \lim_{n\to\infty} \inf f_n(x) &= -\lim_{n\to\infty} \sup(-f_n(x))\end{aligned}$$

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Corollary 9.5 If $\{f_n\}$ is measurable, and $f_n(x)$ converges to $f(x)$ pointwisely a.e., then f is measurable.

Proof. By proposition (9.3), w.l.o.g., $f_n(x)$ conveges to $f(x)$ pointwisely, which follows that

$$f(x) := \lim_{n\to\infty} f_n = \lim_{n\to\infty} \sup f_n(x)$$

i.e., f is measurable due to the measurability of $\lim_{n\to\infty} \sup f_n(x)$. ■

9.5.2. Lebesgue Integration

Definition 9.9 [Simple Function] A function $\phi : \mathbb{R} \to \mathbb{R}$ is **simple** if

- ϕ is measurable and
- $\{\phi(x) \mid x \in \mathbb{R}\}$ takes finitely many values.

More precisely, if the simple function ϕ takes distinct values $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$ on disjoint non-empty sets $A_1, \dots, A_k \subseteq \mathbb{R}$, then

$$\phi = \sum_{i=1}^{k} \alpha_i \mathcal{X}_{A_i}$$

Note that A_i's are measurable since $\phi^{-1}(\{\alpha_i\}) = A_i$

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1. All functions written in the form $\psi = \sum_{i=1}^{\ell} \beta_i \mathcal{X}_{B_i}$, where B_i's are measurable, are simple; All simple functions can be expressed as the form $\psi = \sum_{i=1}^{\ell} \beta_i \mathcal{X}_{B_i}$ (where B_i's are disjoint) uniquely, up to permutation of terms. This is called the canonical form.
2. If ϕ_1, ϕ_2 are simple, then so are

$$\phi_1 + \phi_2, \quad \phi_1 \cdot \phi_2, \quad \alpha \cdot \phi, \max(\phi_1, \phi_2), \quad h \circ \phi.$$

for all function h.

Definition 9.10 [Lebesgue integral for Simple Function] Given a simple function with the canonical form $\phi := \sum_{i=1}^k \alpha_i \mathcal{X}_{A_i}$,

- The Lebesgue integral for ϕ (over $\mathbb{R}$) is

$$\int \phi \, \mathrm{d}m = \sum_{i=1}^k \alpha_i m(A_i),$$

- The Lebesgue integral for ϕ over a measurable set E is

$$\int_E \phi \, \mathrm{d}m = \int \phi \cdot \mathcal{X}_E \, \mathrm{d}m = \sum_{i=1}^k \alpha_i m(A_i \cap E)$$

Proposition 9.7 For any simple function $\phi = \sum_{i=1}^{\ell} \beta_i \mathcal{X}_{B_i}$, where B_i's are not necessarily disjoint, we still have

$$\begin{aligned} \int \phi \, \mathrm{d}m &= \sum_{i=1}^{\ell} \beta_i m(B_i), \\ \int (\phi + \psi) \, \mathrm{d}m &= \int \phi \, \mathrm{d}m + \int \psi \, \mathrm{d}m, \quad \text{where } \psi \text{ is another simple function}, \\ \int \phi \, \mathrm{d}m &\le \int \psi \, \mathrm{d}m, \qquad \text{provided that } \phi \le \psi. \end{aligned}$$

Proof. It suffices to show the first equality. w.l.o.g., suppose $\phi = \beta_1 \mathcal{X}_{B_1} + \beta_2 \mathcal{X}_{B_2}$, which can be reformulated as the canonical form:

$$\phi = (\beta_1 + \beta_2) \mathcal{X}_{B_1 \cap B_2} + \beta_1 \mathcal{X}_{B_1 \cap B_2^c} + \beta_2 \mathcal{X}_{B_1^c \cap B_2}$$

Then we can take the Lebesgue integration for ϕ:

$$\int \phi \, \mathrm{d}m = (\beta_1 + \beta_2) m(B_1 \cap B_2) + \beta_1 m(B_1 \cap B_2^c) + \beta_2 m(B_1^c \cap B_2),$$

which is equal to $\beta_1 m(B_1) + \beta_2 m(B_2)$ due to the caratheodory property (definition (8.2)) ■

Definition 9.11 [Lebesgue integral for Measurable Function] Let f be a measurable function $f:\mathbb{R}\to[0,\infty]$. Then the Lebesgue integral of f is given by:

$$\int f\,\mathrm{d}m = \sup\left\{\int \phi\,\mathrm{d}m \,\middle|\, 0\le\phi\le f, \phi \text{ is simple}\right\} \tag{9.1}$$

We say f is integrable if $\int f\,\mathrm{d}m<\infty$.

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- It's not appropriate if we try to define the Lebesgue integral by

$$\int f\,\mathrm{d}m = \inf\left\{\int \phi\,\mathrm{d}m \,\middle|\, 0\le f\le \phi, \phi \text{ is simple}\right\} \tag{9.2}$$

  The problem is due to the function $f(x)=\frac{1}{\sqrt{x}}$ on $(0,1)$. Note that the function values can be arbitrarily large.

  Since a simple function takes only finitely many values, every simple function that is bounded below by f has to be infinite on a set of non-zero measure.

  Therefore, the integral using your suggested infimum definition would be ∞, whereas the usual Lebesgue integral would have a finite value.

- Also, one can try to define $\int f\,\mathrm{d}m$ for non-measurable function f. The problem is that

$$\int (f+g)\,\mathrm{d}m \ne \int f\,\mathrm{d}m + \int g\,\mathrm{d}m \text{ in general}$$

  We will see the detailed reason later.

Proposition 9.8

- The formula (9.1) and (9.2) matches with each other for any simple functions $\phi\ge 0$.
- For $\alpha\ge 0$,

$$\int \alpha f\,\mathrm{d}m = \alpha\int f\,\mathrm{d}m$$

- If $0 \leq f \leq g$, then

$$\int f \, \mathrm{d}m \leq \int g \, \mathrm{d}m$$

Proof. omitted. ∎

Proposition 9.9 — **Markov Inequality.** Suppose that $f : \mathbb{R} \to [0, \infty]$ is measurable, then

$$m(f^{-1}[\lambda, \infty]) \leq \frac{1}{\lambda} \int f \, \mathrm{d}m$$

Corollary 9.6 If $f : \mathbb{R} \to [0, \infty]$ is integrable, then $m(f^{-1}\{\infty\}) = 0$, i.e., f is finite a.e.

Proof.

$$m(f^{-1}\{\infty\}) \leq m(f^{-1}[\lambda, \infty]) \leq \frac{1}{\lambda} \int f \, \mathrm{d}m, \ \forall \lambda \geq 0.$$

Since $\int f \, \mathrm{d}m$ is finite, we imply $\frac{1}{\lambda} \int f \, \mathrm{d}m$ can be arbitrarily small, i.e., $m(f^{-1}\{\infty\}) = 0$. ∎