9.2. Monday for MAT3006

Reviewing.

- The collection of all Lebesgue measurable subsets, denoted by $\mathcal{M}$, is a σ-algebra
- Borel σ-algebra: the smallest σ-algebra containing all the intervals of $\mathbb{R}$:

  Well-definedness. Let $\mathcal{S} = \{\text{all } \sigma\text{-algebras containing all the inervals of } \mathbb{R}\}$. For example, $\mathbb{P}(R) \in \mathcal{S}$. Note that $\forall f_i \in \mathcal{S}$, $\cap_{i \in I} \mathcal{A}_i \in \mathcal{S}$.

  Then define $\mathcal{B} = \cap_{\mathcal{A} \in \mathcal{S}} \mathcal{A}$, which is the smallest σ-algebra containing all intervals. ■

  Furthermore, $\mathcal{M} \in \mathcal{S}$. Therefore, $\mathcal{B} \subseteq \mathcal{M}$ but they are not equal. (Check Royden's note on blackboard, there exists a counter-example A such that $A \in \mathcal{M}$ but $A \notin \mathcal{B}$.)

- The set $\mathcal{M}$ has a good property: If $N \in \mathcal{M}$ is null, then all $E \subseteq N$ are null sets, and therefore $E \in \mathcal{M}$.

  The probelm is that it is not necessary the case that $N' \in \mathcal{B}$ implies $m\mid_{\mathcal{B}}(N') = 0$, i.e., $E' \in \mathcal{B}, \forall E' \subseteq N'$ does not necessarily hold.

  (check back to the Roydon's counter-example)

- Therefore, we need the **completion process** of $\mathcal{B}$ to get $\mathcal{M}$.

9.2.1. Measurable Functions

Motivation. The Riemann integration divides the function into a grid of 1 (unit) squares, and then measure the altitude of the function at the center of each square. Therefore, the total "volume" of this function is 1 times the sum of the altitudes.

However, the Lebesgue integraion aims to study the vertical length of the function, and the total volumn of this function is 1 times the sum of the vertical lengths. Riemann integration

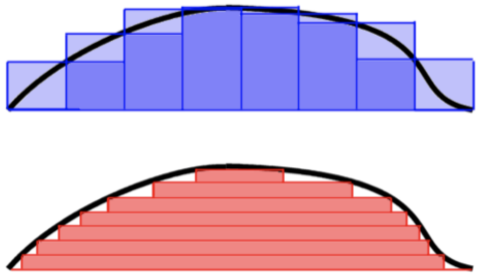

Figure 9.1: Riemann Integration (in blue) and Lebesgue Integration (in red)

Definition 9.1 [Measurable] Let $f : (\mathbb{R}, \mu, m) \to \mathbb{R}$ be a function. We say f is **(Lebesgue) measurable** if $f^{-1}(I) \in \mathcal{M}$ for all intervals $I \subseteq \mathbb{R}$. ∎

Proposition 9.1 If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then f is measurable.

The trick during the proof is to check only intervals of the form (a, ∞) instead of checking all intervals in $\mathbb{R}$.

Proof. 1. By continuity of f, $f^{-1}((a, \infty))$ is open in $\mathbb{R}$.

- Note that any open set U can be expressed as a countable union of open intervals: for given $q \in \mathbb{Q}$, define the set

$$I_q = \bigcup_{I \text{ is an open interval, } q \in I \subseteq U} I,$$

which is a union of non-disjoint open intervals, hence an open interval as well. We claim that $U \subseteq \cup_{q \in \mathbb{Q} \cap U} I_q$:

Consider any $x \in U$. When $x \in \mathbb{Q}$, the result is clear; otherwise there exists an open interval $(x - \varepsilon, x + \varepsilon) \subseteq U$. By the denseness of $\mathbb{Q}$, there exists $q \in \mathbb{Q}$ such that $q \in (x - \varepsilon, x + \varepsilon)$. By definition of I_q, $(x - \varepsilon, x + \varepsilon) \in I_q$. Therefore, $x \in I_q$.

The proof for this statement is complete.

Therefore,

$$f^{-1}((a, \infty)) = \cup_{i=1}^{\infty} U_i \in \mathcal{M}$$

since each open interval $U_i \in \mathcal{M}$

2. For other types of intervals, e.g., $[a,\infty)$, consider

$$\bigcap_{n=1}^{\infty}(a-\frac{1}{n},\infty)=[a,\infty),$$

which follows that

$$f^{-1}([a,\infty))=f^{-1}(\bigcap_{n=1}^{\infty}(a-\frac{1}{n},\infty))=\cap_{n=1}^{\infty}f^{-1}((a-\frac{1}{n},\infty))\in\mathcal{M}$$

The proof for other types of intervals needs similar reformulations of them:

$$f^{-1}((-\infty,a))=f^{-1}(\mathbb{R}\setminus[a,\infty))=\mathbb{R}\setminus f^{-1}([a,\infty))\in\mathcal{M}$$
$$f^{-1}((b,a))=f^{-1}((-\infty,a))\cap f^{-1}((b,\infty))\in\mathcal{M}$$

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1. From the proof above we also find: the function f is measurable if and only if $f^{-1}((a,\infty))\in\mathcal{M}$, for $\forall a\in\mathbb{R}$.
2. Homework question: the function f is measurable if and only if $f^{-1}(B)\in\mathcal{M}$ for $\forall B\in\mathcal{B}$.

Proposition 9.2 1. Constant functions, and monotone functions are measurable

2. If $A\subseteq\mathbb{R}$ is measurable, then the characterstic function

$$\mathcal{X}_A(x):=\begin{cases}1, & \text{if } x\in A\\ 0, & \text{if } x\notin A\end{cases}$$

is measurable.

3. If f is measurable, h is continuous, then $h\circ f$ is continuous.
4. If f,g are measurable, then so is

$$f+g,\quad fg,\quad \max/\min(f,g),\quad |f|$$

Proof.

- (1) and (2) are easy to show.
- The proof for (3) is simply by applying the formula

$$(h \circ f)^{-1}((a,\infty)) = f^{-1}(h^{-1}(a,\infty)),$$

- The proof for the measurability of $f+g$ is by definition:

$$\begin{aligned}(f+g)^{-1}(a,\infty) &= \{x \mid f+g \in (a,\infty)\} \\ &= \cup_{q\in\mathbb{Z}}(\{x \mid f \in (q,\infty)\} \cap \{x \mid g \in (a-q,\infty)\}) \\ &= \cup_{q\in\mathbb{Z}}(f^{-1}(q,\infty) \cap f^{-1}(a-q,\infty)) \in \mathcal{M}\end{aligned}$$

The measurability of $fg, |f|, \max / \min(f,g)$ are by the equalities

$$\begin{aligned}fg &= \frac{1}{4}[(f+g)^2 + (f-g)^2] \\ |f| &= h \circ f \quad h(x) = |x| \\ \max / \min(f,g) &= \frac{1}{2}(f+g \pm |f-g|)\end{aligned}$$

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(R) If both f,g are measurable, then $g \circ f$ is not necessarily measurable.

Definition 9.2 [Almost Everywhere] Let $f,g : (\mathbb{R},\mu,m) \to \mathbb{R}$. We say $f = g$ almost everywhere (a.e.) if $E := \{x \mid f(x) \neq g(x)\}$ is a null set.

More generally, we say $f(x)$satisfies a condition on (R,μ,m) a.e. if the set

$$\{x \mid f(x) \text{ does not satisfy the condition}\} \text{ is a null set.}$$

For example, the characteristic function $\mathcal{X}_{\mathbb{Q}}(x)$ is equal to zero function a.e.

The measurability ignores the null set.

Proposition 9.3 Suppose that f is measurable, and $g = f$ a.e., then g is measurable.

Proof. Note that

$$g^{-1}((a,\infty)) = \{x \mid g(x) \in (a,\infty), g(x) = f(x)\} \cup \{x \mid g(x) \in (a,\infty), g(x) \neq f(x)\}$$

where $\{x \mid g(x) \in (a,\infty), g(x) \neq f(x)\} \subseteq E$, i.e., belongs to $\mathcal{M}$; and

$$\{x \mid g(x) \in (a,\infty), g(x) = f(x)\} = f^{-1}((a,\infty)) \cap E^c \in \mathcal{M}.$$

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(R) During the proof, we have used the fact that $N \subseteq E$ is measurable for all null set E.

Definition 9.3 [Measurable on extended real line] A function $f : \mathbb{R} \to [-\infty, \infty]$ is measurable if

$$f^{-1}((a,\infty]) \in \mathcal{M}, \quad \forall a \in \mathbb{R}.$$

Or equivalently,

$$f^{-1}(B) \in \mathcal{M}, \forall B \in \mathcal{B}, \text{ and } f^{-1}(\{-\infty\}), f^{-1}(\{\infty\}) \in \mathcal{M}.$$

Example:

$$f(x) = \begin{cases} \tan x & x \neq \frac{2n+1}{2}\pi, n \in \mathbb{Z} \\ \infty, & x = \frac{2n+1}{2}\pi, n \in \mathbb{Z} \end{cases}$$

is measurable.