8.4. Wednesday for MAT3006

Reviewing.

- All null sets are measurable
- If $E \subseteq \mathbb{R}$ is measurable, then $E^c := R \setminus E$ is measurable.
- E_i is measurable implies $\cup_{i=1}^{n} E_i$ is measurable.

8.4.1. Remarks on Lebesgue Measurability

Proposition 8.5 If E_i is measurable for $\forall i \in \mathbb{N}$, then so is $\cup_{i=1}^{\infty} E_i$. Moreover, if further E_i's are pairwise disjoint, then

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i)$$

Proof. • Consider the case where E_i's are measurable, pairwise disjoint first. For all subsets $A \subseteq \mathbb{R}$, and all $n \in \mathbb{N}$,

$$\begin{aligned} m^*(A) &= m^*(A \cap (\cup_{i=1}^{n} E_i)) + m^*(A \cap (\cup_{i=1}^{n} E_i)^c) && (8.4a) \\ &= [m^*(A \cap (\cup_{i=1}^{n} E_i) \cap E_n) + m^*(A \cap (\cup_{i=1}^{n} E_i) \cap E_n^c)] + m^*(A \cap (\cup_{i=1}^{n} E_i)^c) && (8.4b) \\ &= [m^*(A \cap E_n) + m^*(A \cap (\cup_{i=1}^{n-1} E_i)] + m^*(A \cap (\cup_{i=1}^{n} E_i)^c) && (8.4c) \end{aligned}$$

where (8.4a) is by the measurability of $\cup_{i=1}^{n} E_i$; (8.4b) is by the measurability of E_n; (8.4c) is by direct calculation.

Proceeding these trick similarly, we obtain:

$$\begin{aligned} m^*(A) &= [m^*(A \cap E_n) + m^*(A \cap (\cup_{i=1}^{n} E_i)^c)] + m^*(A \cap (\cup_{i=1}^{n-1} E_i)] && (8.5a) \\ &= \sum_{\ell=1}^{n} m^*(A \cap E_\ell) + m^*(A \cap (\cup_{i=1}^{\ell} E_i)^c) && (8.5b) \\ &\geq \sum_{\ell=1}^{n} m^*(A \cap E_\ell) + m^*(A \cap (\cup_{i=1}^{\infty} E_i)^c) && (8.5c) \end{aligned}$$

for any $n \in \mathbb{N}$, where (8.5c) is by lower bounding $(\cup_{i=1}^{\ell} E_i)^c \supseteq (\cup_{i=1}^{\infty} E_i)^c$. Taking $n \to \infty$ in (8.5c), we imply

$$m^*(A) \geq \sum_{\ell=1}^{\infty} m^*(A \cap E_\ell) + m^*(A \cap (\cup_{i=1}^{\infty} E_i)^c) \tag{8.5d}$$

$$\geq m^*(\cup_{i=1}^{\infty}(A \cap E_i)) + m^*(A \cap (\cup_{i=1}^{\infty} E_i)^c) \tag{8.5e}$$

$$= m^*(A \cap (\cup_{i=1}^{\infty} E_i)) + m^*(A \cap (\cup_{i=1}^{\infty} E_i)^c) \tag{8.5f}$$

where (8.5e) is by the countable sub-addictivity of m^*. Therefore, $\cup_{i=1}^{\infty} E_i$ is measurable.

- Moreover, taking $A = \cup_{i=1}^{\infty} E_i$ in (8.5d) gives

$$m^*(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(E_i) + m^*(\emptyset) = \sum_{i=1}^{\infty} m^*(E_i) + 0.$$

- Now suppose that E_i's are measurable but not necessarily pairwise disjoint. We need to show $\bigcup_{i=1}^{\infty} E_i$ is measurable. The way is to construct the disjoint sequence of sets first:

$$\begin{cases} F_1 = E_1, \\ F_{k+1} = E_k \setminus \left(\cup_{i=1}^{k} E_i\right), \ \forall k > 1 \end{cases} \implies \cup_{i=1}^{\infty} F_i = \cup_{i=1}^{\infty} E_i$$

It's clear that F_i's are pairwise disjoint and measurable, which implies $\cup_{i=1}^{\infty} E_i = \cup_{i=1}^{\infty} F_i$ is measrable. The proof is complete. ■

Notations. We denote $\mathcal{M}$ as the collection of all **(Lebesgue) measurable** subsets of $\mathbb{R}$, and

$$m(E) = m^*(E), \quad \forall E \in \mathcal{M}$$

8.4.2. Measures In Probability Theory

Definition 8.8 [σ-Algebra]

- Let Ω be any set, and $\mathbb{P}(\Omega)$ (**power set**) denotes the collection of all subsets of Ω
- A family of subsets of Ω, denoted as $\mathcal{T}$, is a σ-algebra if it satisfies
  1. $\emptyset, \Omega \in \mathcal{T}$
  2. If $E_i \in \mathcal{T}$ for $\forall i \in \mathbb{N}$, then $\cup_{i=1}^{\infty} E_i \in \mathcal{T}$ (and therefore $\cap_{i=1}^{\infty} E_i \in \mathcal{T}$).

Definition 8.9 [Measure] A **measure** on a σ-algerba $(\Omega, \mathcal{T})$ is a function $\mu : \mathcal{T} \to [0, \infty]$ such that

- $\mu(\emptyset) = 0$
- $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ whenever E_i's are pairwise disjoint in $\mathcal{T}$.

As a result, $(\Omega, \mathcal{T}, \mu)$ is called a **measurable space**.

Example 8.5 1. Let Ω be any set, $\mathcal{T} = \mathcal{M}$, and $\mu(E) = |E|$ (the number of elements in E). Then $(\Omega, \mathbb{P}(\Omega), \mu)$ is a measure space, and μ is called a counting measure on Ω.

Definition 8.10 [Borel σ-algebra] Let $\boldsymbol{B}$ be a collection of all intervals in $\mathbb{R}$. Then there is a **unique** σ-algebra $\mathcal{B}$ of $\mathbb{R}$, such that

1. $\boldsymbol{B} \subseteq \mathcal{B}$
2. For all σ-algebra $\mathcal{T}$ containing $\boldsymbol{B}$, we have $\mathcal{B} \subseteq \mathcal{T}$

This $\mathcal{B}$ is called a **Borel σ-algebra**

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1. In particular, $C_i \in \mathcal{B}$ implies $\cup_{i=1}^{\infty} C_i$ and $\cap_{i=1}^{\infty} C_i \in \mathcal{B}$.

2. $\mathcal{B} \subseteq \mathcal{M}$, since $\boldsymbol{B} \subseteq \mathcal{M}$ and $\mathcal{M}$ is a σ-algebra.

3. However, $\mathcal{M}$ and $\mathcal{B}$ are not equal. The element $C \in \mathcal{B}$ is called **Borel measurable subsets**

Definition 8.11 [complete] Let $(\Omega, \mathcal{T}, \mu)$ be a measurable space. Then we say it is **complete** if for any $E \in \mathcal{T}$ with $\mu(E) = 0$, $N \subseteq E$ implies $N \in \mathcal{T}$. (and therefore $\mu(N) = 0$)

■ **Example 8.6** 1. $(\mathbb{R}, \mu, m^*)$ is complete.

Reason: if $m^*(E) = 0$, then $m^*(N) = 0$, $\forall N \subseteq E$

2. $(\mathbb{R}, \mu, m)$ is complete.

Reason: the same as in (1)

3. However, $(\mathbb{R}, \mathcal{B}, m \mid_{\mathcal{B}})$ is not complete. (left as exercise)

Then we study the difference between $\mathcal{B}$ and $\mathcal{M}$:

Definition 8.12 [Completion] Let $(\Omega, \mathcal{T}, \mu)$ be measurable space. The **completion** of $(\Omega, \mathcal{T}, \mu)$ with respect to μ is the smallest complete σ-algebra containing $\mathcal{T}$, denoted as $\overline{\mathcal{T}}$. More precisely,

$$\overline{\mathcal{T}} = \{G \cup N \mid G \in \mathcal{T}, N \subseteq F \in \mathcal{T}, \text{with } \mu(F) = 0\}$$

e.g., take $G = \emptyset \in \mathcal{T}$. For all $F \in \mathcal{T}$ such that $\mu(F) = 0$, $N \subseteq F$ implies $N \in \overline{\mathcal{T}}$.

R If further define $\overline{\mu} : \overline{\mathcal{T}} \to [0, \infty]$ by

$$\overline{\mu}(G \cup N) = \mu(G),$$

then $(\Omega, \overline{\mathcal{T}}, \overline{\mu})$ is a measurable space.

Theorem 8.3 The completion of $(\mathbb{R}, \mathcal{B}, m|_{\mathcal{B}})$ is $(\mathbb{R}, \mathcal{M}, m)$

R Another completion of $(\mathbb{R}, \mu, m)$ is as follows:

Define $\ell(\{a,b\}) = b - a$ for all intervals $\{a,b\} \in \boldsymbol{B}$ Then by Caratheodory extension theorem, we can extend $\ell : \boldsymbol{B} \to [0,\infty]$ to $\ell : \mathcal{B} \to [0,\infty]$.

Complete $\ell : \mathcal{B} \to [0,\infty]$ to $\bar{\ell} : \mathcal{M} \to [0,\infty]$. Then $\bar{\ell} = m$ as in our course.