8.2. Monday for MAT3006

Reviewing. We define the **outer** measure of a subset $E \subseteq \mathbb{R}$ to be

$$m^*(E) = \inf\left\{\sum_{n=1}^{\infty} m(I_n) \middle| E \subseteq \bigcup_{n=1}^{\infty} I_n,\ I_n\text{'s are open intervals}\right\}$$

One Special Property of Outer Measure:

$$m^*(\cup_{n=1}^{\infty} E_n) \le \sum_{n=1}^{\infty} m^*(E_n)$$

8.2.1. Remarks for Outer Measure

We want to make a special hyphothesis become true: If E_n's are disjoint, then

$$m^*(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m^*(E_n) \tag{8.3}$$

However, (8.3) does not necessary hold for a sequence of disjoint subsets $\{E_n\}$. One counter-example is shown in Example (8.2).

■ **Example 8.2** [Vitali Set] Suppose that $A \subseteq [0,1]$ satisfies the following properties:

- For any $x \in \mathbb{R}$, there exists $q \in \mathbb{Q}$ such that $x + q \in A$.
- If $x, y \in A$ such that $x \ne y$, then $x - y \notin \mathbb{Q}$

In other words, the group $\mathbb{R}$ is partitioned into the cosets of its addictive subgroup $\mathbb{Q}$, and the properties above say that A contains exactly one member of each coset of $\mathbb{Q}$. The existence of such A relies on the Axiom of Choice. Moreover, we imply:

- $[0,1] \subseteq \bigcup_{q \in [-1,1] \cap \mathbb{Q}} (A - q)$: since $\forall x \in [0,1]$, there exists $q \in \mathbb{Q}$ s.t. $x + q \in A$, which implies $x \in A = q$. Moreover, we can bound the possible region of q:

$$0 \le x + q \le 1 \implies -x \le q \le 1 - x \implies -1 \le q \le 1$$

- $\bigcup_{q\in[-1,1]\cap\mathbb{Q}}(A-q)\subseteq[-1,2]$: elements in $A-q$ are of the form $x-q, x\in[0,1], q\in[-1,1]$, and therefore $x-q\in[-1,2]$.
- The sets $(A-q)$ are disjoint as q varies, i.e., $(A-q_1)\cap(A-q_2)=\emptyset, \forall q_1\neq q_2\in[-1,1]\cap\mathbb{Q}$: Suppose on the contrary that there exists $y\in(A-q_1)\cap(A-q_2)$, which follows

$$y+q_1, y+q_2\in A, y+q_1\neq y+q_2 \implies (y+q_1)-(y+q_2)=q_1-q_2\notin\mathbb{Q}$$

Suppose on the contrary that (8.3) holds for $\{A-q \mid \forall q\in[-1,1]\cap\mathbb{Q}\}$, then

$$m^*\left(\bigcup_{q\in[-1,1]\cap\mathbb{Q}}(A-q)\right)=\sum_{q\in[-1,1]\cap\mathbb{Q}}m^*(A-q)=\sum_{q\in[-1,1]\cap\mathbb{Q}}m^*(A), \tag{8.4}$$

where the second equality is because that $m^*(A-q)=m^*(A), \forall q$. However,

$$1=m^*([0,1])\leq m^*\left(\bigcup_{q\in[-1,1]\cap\mathbb{Q}}(A-q)\right)\leq m^*([-1,2])=3 \tag{8.5}$$

From (8.4) we derive the $m^*\left(\bigcup_{q\in[-1,1]\cap\mathbb{Q}}(A-q)\right)$ can either be 0 or ∞, which is a contradiction. ■

8.2.2. Lebesgue Measurable

Therefore, (8.5) does not hold for some bad subsets of $\mathbb{R}$, which are sets cannot be explicitly described. Let's focus on sets with good behaviour only:

Definition 8.2 [Carathedory Property] A subset $E\subseteq\mathbb{R}$ is **measurable** if

$$m^*(A)=m^*(A\cap E)+m^*(A\setminus E) \tag{8.6}$$

for all subsets $A\subseteq\mathbb{R}$, where E is not assumed to be in A, i.e., $A\setminus E:=A\cap E^c$. ■

R The equality (8.6) holds if we can show $m^*(A)\geq m^*(A\cap E)+m^*(A\setminus E)$. There are many other equivalent definitions for measurable set $E\subseteq\mathbb{R}$:

1. For any $\varepsilon > 0$, there exists open set $U \supseteq E$ such that

$$m^*(U \setminus E) \leq \varepsilon$$

2. Its outer and inner measures are equal:

$$m^*(E) = m_*(E) := \sup \left\{ \sum_{n=1}^{\infty} m(I_n) \middle| \bigcup_{n=1}^{\infty} I_n \subseteq E,\ I_n\text{'s are compact, and disjoint subsets} \right\}$$

Note that the inner measure m_* admits the inequality

$$m_*(\cup_{n=1}^{\infty} E_n) \geq \sum_{n=1}^{\infty} m_*(E_n),\ \text{for disjoint } E_n$$

R If $E \subseteq \mathbb{R}$, then for all $B \supseteq E$, we have

$$m^*(B) = m^*(B \cap E) + m^*(B \setminus E) = m^*(E) + m^*(B \setminus E): \tag{8.7}$$

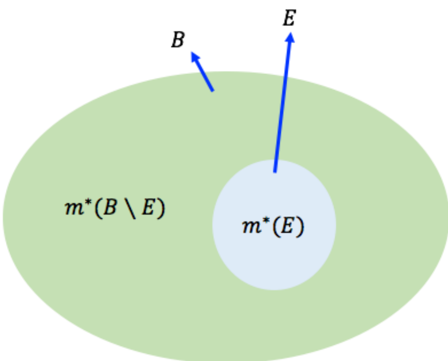

Figure 8.1: Illustration for the useful equality (8.7)

Proposition 8.3 1. If $E \subseteq \mathbb{R}$ is null, then E is measurable

2. If I is any interval, then I is measurable
3. If E is measurable, then $E^c := \mathbb{R} \setminus E$ is measurable
4. If E is measurable, then both $\cup_{i=1}^{n} E_i$ and $\cap_{i=1}^{n} E_i$ are measurable

Proof. 1. For any subsets A,

$$\begin{cases} m^*(A \cap E) = 0 \\ m^*(A \cap E^c) \leq m^*(A) \end{cases} \implies m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c).$$

2. Take $I = [a,b]$. For all $A \subseteq \mathbb{R}$,

   - take $\{I_n\}$ such that $A \subseteq \cup_{n=1}^{\infty} I_n$ and

$$\sum_{n=1}^{\infty} m^*(I_n) \leq m^*(A) + \varepsilon \tag{8.8}$$

   - Note that the $m^*(A \cap I)$ can be upper bounded:

$$A \cap I \subseteq \cup_{n=1}^{\infty}(I_n \cap I) \implies m^*(A \cap I) \leq \sum_{n=1}^{\infty} m^*(I_n \cap [a,b])$$

   Similarly, $m^*(A \cap I^c)$ can be upper bounded:

$$A \cap I^c \subseteq \cup_{n=1}^{\infty} I_n \cap ((-\infty,a) \cup (b,\infty)) = \left(\bigcup_{n=1}^{\infty} I_n \cap (-\infty,a)\right) \cup \left(\bigcup_{n=1}^{\infty} I_n \cap (b,\infty)\right),$$

   i.e.,

$$m^*(A \cap I^c) \leq \sum_{n=1}^{\infty} m^*(I_n \cap (-\infty,a)) + m^*(I_n \cap (b,\infty))$$

   - Therefore,

$$\begin{aligned} m^*(A \cap I) + m^*(A \cap I^c) &\leq \sum_{n=1}^{\infty} m^*(I_n \cap (-\infty,a)) + m^*(I_n \cap [a,b]) + m^*(I_n \cap (b,\infty)) \\ &= \sum_{n=1}^{\infty} m^*(I_n \cap (-\infty,\infty)) = \sum_{n=1}^{\infty} m^*(I_n) \\ &\leq m^*(A) + \varepsilon, \end{aligned}$$

   i.e., $m^*(A \cap I) + m^*(A \cap I^c) \leq m^*(A)$.

3. Part (3) is trivial.

4. Part (4) is by induction on n: suppose that

   - E_i is measurable for $i = 1, \ldots, k+1$

- $E = \cup_{i=1}^{k} E_i$ is measurable

By the measurablitiy of E_{k+1},

$$m^*(A \cap E^c) = m^*(A \cap E^c \cap E_{k+1}) + m^*(A \cap E^c \cap E_{k+1}^c) \tag{8.9}$$

By the measurablitiy of E,

$$\begin{aligned} m^*(A) &\geq m^*(A \cap E) + m^*(A \cap E^c) \\ &\geq [m^*(A \cap E) + m^*(A \cap E^c \cap E_{k+1})] + m^*(A \cap E^c \cap E_{k+1}^c) \end{aligned} \tag{8.10}$$

It's easy to show

$$E \cup (E^c \cap E_{k+1}) = E \cup E_{k+1},$$

which implies

$$\begin{aligned} m^*(A \cap (E \cup E_{k+1})) &= m^*(A \cap (E \cup (E^c \cap E_{k+1}))) \\ &= m^*((A \cap E) \cup (A \cap (E^c \cap E_{k+1}))) \\ &\leq m^*(A \cap E) + m^*(A \cap (E^c \cap E_{k+1})) \end{aligned} \tag{8.11}$$

Substituting (8.11) into (8.10) gives

$$m^*(A) \geq m^*(A \cap (E \cup E_{k+1})) + m^*(A \cap (E \cup E_{k+1})^c),$$

i.e., $E \cup E_{k+1}$ is measurable as well.

By the equality

$$\mathbb{R} \setminus \left(\bigcup_{i=1}^{n} E_i \right) = \bigcup_{i=1}^{n} (\mathbb{R} \setminus E_i),$$

and the result in part (3), one can show $\cap_{i=1}^{n} E_i$ is measurable as well.

■

Proposition 8.4 If E_i is measurable, then $\cup_{i=1}^{\infty} E_i$ is measurable. Moreover, if E_i's are disjoint, then

$$m^*(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(E_i)$$

Note that $m^*(A) = 0$ for Vitali set A: suppose contrary that $m^*(A) = 0$, i.e., A is null set. Since countably null set is also measurable, together with (8.4), we imply

$$m^*\left(\bigcup_{q\in[-1,1]\cap\mathbb{Q}} (A-q)\right) = 0,$$

which contradicts to (8.5).

Notations.

1. We will write $m(E) = m^*(E)$ for all measurable sets $E \subseteq \mathbb{R}$, and therefore

$$m(\{a,b\}) = m^*(\{a,b\}) = b - a$$

2. The sets E satisfying

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

are called **Lebesgue** measurable in some other textbooks.