7.2. Monday for MAT3006

Our first mid-term will be held on this Wednesday.

Reviewing.

- We constructed a kind of function to measure the length of a given subset $E \subseteq \mathbb{R}$:

$$m^*(E) = \inf \left\{ \sum_{n=1}^{\infty} m(I_n) \middle| E \subseteq \bigcup_{n=1}^{\infty} I_n, \ I_n \text{ are open intervals} \right\}$$

  which is called the **outer measure**

7.2.1. Remarks on the outer measure

Proposition 7.8

1. $m^*(\phi) = 0$, $m^*(\{x\}) = 0$.
2. $m^*(E + x) = m^*(E)$
3. $m^*(I) = b - a$, where I denotes any interval with endpoints a or b.
4. If $A \subseteq B$, then $m^*(A) \leq m^*(B)$
5. $m^*(kE) = |k| m^*(E)$
6. $m^*(\cup_{m=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} m^*(E_n)$ for subsets $E_n \subseteq \mathbb{R}$

R The trick in the proof to show $x \leq y$ is by the argument $x \leq y + \varepsilon, \forall \varepsilon > 0$.

(1),(2),(5) is clear. (4) is by one-line argument:

Suppose that $B \subseteq \cup_{n=1}^{\infty} I_n$, then $A \subseteq \cup_{n=1}^{\infty} I_n$.

Proof for (3). Consider $m^*([a,b])$ first. The proof for $m^*([a,b]) \leq b - a$ is by explicitly constructing a sequence of open intervals:

$$[a,b] \subseteq (a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}) \cup (a,a) \cup \cdots$$

It follows that

$$m^*([a,b]) \leq m((a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2})) + 0 + \cdots + 0$$
$$= (b-a) + \varepsilon, \ \forall \varepsilon > 0$$

In particular, $m^*([a,b]) \leq b - a$.

Conversely, the proof for $b - a \leq m^*([a,b])$ is by implicitly constructing a sequence of open interval via the infimum. For all $\varepsilon > 0$, there exists $I_n,\ n \in \mathbb{N}$ such that

$$[a,b] \subseteq \cup_{n=1}^{\infty} I_n, \qquad \sum_{n=1}^{\infty} m(I_n) \leq m^*([a,b]) + \varepsilon.$$

By Heine-Borel Theorem, there exists finite subcover $[a,b] \subseteq \cup_{n=1}^{k} I_n$. Let $I_n = (\alpha_n, \beta_n)$, consider $\alpha := \min\{\alpha_n \mid a \in I_n\}$ and $\beta := \max\{\beta_n \mid b \in I_n\}$. Then we imply

$$[a,b] \subseteq (\alpha, \beta) \subseteq \cup_{n=1}^{k} I_n.$$

It's clear that $\beta - \alpha \leq \sum_{n=1}^{k} m(I_n)$, which follows that

$$b - a \leq \beta - \alpha \leq \sum_{n=1}^{k} m(I_n) \leq \sum_{n=1}^{\infty} m(I_n) \leq m^*([a,b]) + \varepsilon$$

The proof is complete.

The other cases of (3) follows similarly. For example, $m^*((a,b))$ can be lower bounded as:

$$m^*((a,b)) + \varepsilon \geq m^*([a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2}]) + \varepsilon = a - b$$

■

Proof for (6). The case for which $m^*(E_n) = \infty$ for some n is trivial, since both sides clearly equal to infinite. Consider the case where $m^*(E_n) < \infty$ only.

By definition, for each E_n we can find $\{I_{n,k}\}_{k=1}^{\infty}$ such that

$$E_n \subseteq \cup_{k=1}^{\infty} I_{n,k}, \quad \sum_{k=1}^{\infty} m(I_{n,k}) \leq m^*(E_n) + \tfrac{\varepsilon}{2^n}.$$

It follows that

- $\cup_{n=1}^{\infty} \cup_{k=1}^{\infty} I_{n,k}$ is a countable open cover of $\cup_{n=1}^{\infty} E_n$, i.e.,

$$m^*(\cup_{n=1}^{\infty} E_n) \leq \sum_{n,k} m(I_{n,k})$$

- $$\sum_{n,k} m(I_{n,k}) \leq \sum_{n=1}^{\infty} m^*(E_n) + \varepsilon$$

The proof is complete. ■

The natural question is that when does the equality in (6) holds? We will study it in next week.

Definition 7.4 [Null Set] The set $E \subseteq \mathbb{R}$ is a **null set** if $m^*(E) = 0$. ▪

Null sets are the set of points which we can "ignore" when consider the length for sets.

Corollary 7.1
1. If E is null, so is any subset $E' \subseteq E$
2. If E_n is null for all $n \in \mathbb{E}$, so is $\cup_{n=1}^{\infty} E_n$
3. All countable subsets of $\mathbb{R}$ are null.

Proof. (1) follows from (4) in proposition (7.8); (2) follows from (6) in proposition (7.8); (3) follows from (1) and (6) in proposition (7.8). ■

In the remaining of this lecture let's discuss two interesting questions:

1. Are there any uncountable null sets?
2. Both "null" and "meagre" is small. Is null = meagre?

The classic example, cantor set is meagre, null, and uncountable:

▪ **Example 7.3** [Cantor Set] Starting from the interval $C_0 = [0,1]$, one delete the open middle third $(1/3, 2/3)$ from C_0, leaving two line segments:

$$C_1 = [0,1/3] \cup [2/3,1].$$

Next, the open middle third of each of these remaining segments is deleted, leaving four line segments:

$$C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1].$$

Continuing this process infinitely, and define $C = \cap_{n=1}^{\infty} C_n$.

1. The cantor set C is null, since $C \subseteq C_n$ for all n, i.e.,

$$m^*(C) \le m^*(C_n) = (2/3)^n, \ \forall n \implies m^*(C) = 0.$$

2. The cantor set C is uncountable: every element in C can be expressed uniquely in ternary expression, i.e., only use 0,1,2 as digits. Suppose on the contrary that C is countable, i.e., $C = \{c_n\}_{n\in\mathbb{N}}$. Then construct a new number such that $c \notin \{c_n\}_{n\in\mathbb{N}}$ by diagonal argument.
3. C is nowhere dense, i.e., C is meagre:
   (a) Firstly, C is closed, since intersection of closed sets is closed.
   (b) Suppose on the contrary that $(\alpha,\beta) \subseteq C$ for some open interval (α,β), then $(\alpha,\beta) \subseteq C_n = \sqcup_{k=1}^{2^n}[a_{n,k}, b_{n,k}]$ for all n. Therefore, for any fixed n, $(\alpha,\beta) \subseteq [a_{n,k}, b_{n,k}]$ for some k, which implies

$$\beta - \alpha < b_{n,k} - a_{n,k} = \frac{1}{3^n}, \ \forall n \in \mathbb{N}$$

   Therefore, $\beta - \alpha = 0$, which is a contradiction.

■

R However, the answer for the second question is no. There exists a mergre set S with $m^*(S) = \infty$; and also a null set that is co-meagre. The construction of these examples are left as exercise.

The outer measure m^* is a special measure of the length of a given subset. Now we define the generalized measure of length:

Definition 7.5 [Measure] A meaasure of length for all subsets in $\mathbb{R}$ is a function m satisfying

1. $m(\emptyset) = m(\{x\}) = 0$
2. $m(\{a,b\}) = b - a$
3. $m(A + x) = m(A), \forall x \in \mathbb{R}$
4. If $A \subseteq B$, then $m(A) \leq m(B)$
5. $m(kA) = |k| m(A)$
6. If $E_i \cap E_j = \emptyset, \forall i \neq j$, then

$$\sum_{i=1}^{\infty} m(E_i) = m(\cup_{i=1}^{\infty} E_i)$$

Question: m^* satisfies (1) to (5), does m^* satisfies (6) for any subsets? In other words, is outer measure the special case of the definition of measure?

Answer: no.