6.2. Monday for MAT3006

6.2.1. Compactness in Functional Space

In functional space, previous study have shown that closedness and boundedness is not equivalent to compactness. We need the equi-continuity to rescue the situation:

Definition 6.2 [Equi-continuity] Let $X \subseteq \mathbb{R}^n$. A subset $\mathcal{T} \subseteq \mathcal{C}(X)$ is called **equi-continuous** if for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $d(x,y) < \delta, x, y \in X$

$$d_\infty(f(x), f(y)) < \varepsilon, \forall f \in \mathcal{T}$$

■ **Example 6.3** 1. Let $\mathcal{T}$ be a collection of Lipschitz continuous functions with the same Lipschitz constant L, i.e., $\forall f \in \mathcal{T}$, $|f(x) - f(y)| < L|x-y|$ for $\forall x, y \in X$. It's clear that $\mathcal{T}$ is equi-continuous.

2. Let $\mathcal{T} \subseteq \mathcal{C}[a,b]$ be such that

$$\sup_{x \in [a,b]} |f'(x)| < M, \quad \forall f \in \mathcal{T},$$

then for any $\forall x, y \in [a,b]$, we imply $|f(y) - f(x)| = |f'(\xi)||y-x|$ for some $\xi \in [a,b]$. Therefore,

$$|f(y) - f(x)| < M|y-x|, \quad \forall f \in \mathcal{T},$$

i.e., $\mathcal{T}$ reduces to the space studied in (1) with Lipschitz constant M, thus is equi-continuous.

Theorem 6.3 Let $K \subseteq \mathbb{R}^n$ be a compact set, and $\mathcal{T} \subseteq \mathcal{C}(K)$. Then $\mathcal{T}$ is **compact** if and only if $\mathcal{T}$ is **closed**, **uniformly bounded**, and **equicontinuous**.

Proof. To be added. ■

Corollary 6.1 Let $K \subseteq \mathbb{R}^n$ be compact, and $\{f_n\}$ be a sequence of uniformly bounded, equi-continuous functions on K. Then $\{f_n\}$ has the **Bolzano-Weierstrass property**, i.e., it has a convergent subsequence.

Proof. To be added. ■

6.2.2. An Application of Ascoli-Arzela Theorem

The Ascoli-Arzela Theorem has a novel application on the ODE. Consider the IVP problem again:

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \\ y(\alpha) = \beta \end{cases} \tag{6.2}$$

where f is continuous on a rectangle R containing (α, β). Now we show the existence of Picard-Lindelof Theorem without the Lipschitz condition:

Theorem 6.4 — **Cauchy-Peano Theorem.** Consider the problem (6.2). Then there exists a solution of this ODE on some rectangle $R' \subseteq R$.

Proof. To be added. ■