5.5. Wednesday for MAT3006

5.5.1. Remarks on Baire Category Theorem

Theorem 5.4 — Baire Category Theorem. If (X,d) is complete, and $E_i \subseteq X$ is nowhere dense for $i \in \mathbb{N}$, then
$$\bigcup_{i=1}^{\infty} \overline{E_i}$$
contains no open balls.

Definition 5.4 Let (X,d) be a complete metric space.

1. We say $S \subseteq X$ is **meager** if
$$S = \bigcup_{i=1}^{\infty} E_i, \qquad E_i \text{ is nowhere dense}$$
In this case we say S is of **first category**.
2. $S' \subseteq X$ is **comeager** if
$$S' = X \setminus S, \qquad \text{where } S \text{ is meager}$$

For example, $\mathbb{Q} = \cup_{x\in\mathbb{Q}}\{x\}$ is megre; $\mathbb{R} \setminus \mathbb{Q}$ is comeager.

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1. By the Baire Category Theorem, $\cup_{i=1}^{n} \bar{E}_i$ contains no open balls, i.e.,
$$S := \cup_{i=1}^{\infty} E_i \subseteq \cup_{i=1}^{\infty} \bar{E}_i$$
contains no open balls.
2. S' is comeager implies S' is dense in X: for $\forall x \in X$, $B_{1/n}(x) \cap S'$ is non-empty, since otherwise $X \setminus S'$ contains a open ball, which is a contradiction. Therefore, $x \in \overline{S'}$.

Proposition 5.7 If a set S is meager, it cannot be comeager and vice versa.

Proof. Suppose on contrary that S is meager and comeager, then

$$S = \bigcup_{i=1}^{\infty} E_i, \quad E_i \text{ is nowhere dense}$$
$$X \setminus S = \bigcup_{j=1}^{\infty} F_j, \quad F_j \text{ is nowhere dense}$$

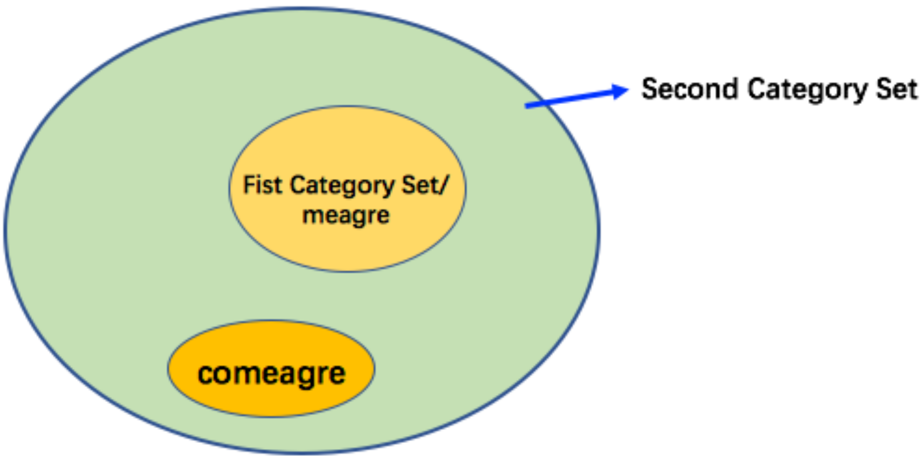
Therefore,

$$X = \bigcup_{i=1}^{\infty} E_i \cup \bigcup_{j=1}^{\infty} F_j$$

is a countable union of nowhere desne sets. By applying Baire Category Theorem, X has no open balls, which is a contradiction. ∎

R We say $S \subseteq X$ is of **first category** if S is meager. Any subset that is not of first category is of **second category**. Therefore, comeager implies second category.

We illustrate the relationship above in the figure below:



Note that there are subsets that are **neither meager nor co-meager**.

■ **Example 5.6** 1. Here is another proof of $[0,1]$ is un-countable: Suppose on the contrary that $[0,1]$ is countable, then we imply

$$[0,1] = \bigcup_{n\in\mathbb{N}} \{x_n\}, \quad \text{for some } x_n.$$

Applying Baire Category Theorem (since $[0,1]$ is complete), $[0,1] = \cup_{n\in\mathbb{N}}\{x_n\}$ contains no open balls. However, the open ball $(0.5, 0.7) \subseteq [0,1]$, which is a contradiction.

2. The set $X := \mathcal{C}[a,b]$ is complete.

   (a) The set of all nowhere differentiable functions is **of 2nd Category** in $\mathcal{C}[a,b]$. (Check Theorem (4.1) in MAT2006) Actually, the set of all nowhere differentiable functions is **comeager**. The proof for this statement is omitted.

   (b) Due to the relationship

$$\mathcal{P}[a,b] \subseteq \mathcal{C}^{\infty}[a,b] \subseteq \{f : [a,b] \to \mathbb{R} \mid f \text{ is differentiable somewhere}\}$$

   and that the last subset is meager, we imply $\mathcal{P}[a,b]$ and $\mathcal{C}^{\infty}[a,b]$ is meager.

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5.5.2. Compact subsets of $\mathcal{C}[a,b]$

Recall that for metrice spaces, the compactness implies closed and bounded, but in general the converse does not hold. We will study extra conditions to make subsets of $\mathcal{C}[a,b]$ compact.

Definition 5.5 [(Uniformly) Bounded] The subset S in metric space $(\mathcal{C}[a,b], d_{\infty})$ is **(uniformly) bounded** if there exists $M > 0$ such that

$$\sup_{f\in S} \|f\|_{\infty} = M$$

■

In next class, we will show that $K \subseteq \mathcal{C}[a,b]$ is compact if and only if K is closed,(uniformly) bounded, and equi-continuous.