5.2. Monday for MAT3006

Our first quiz will be held on this Wednesday.

Reviewing. We have shown that the algebra $\mathcal{A} \subseteq \mathcal{C}(X)$ with separation, non-vanishing property implies $\overline{\mathcal{A}} = \mathcal{C}(X)$.

Now we show that if $\overline{\mathcal{A}} = \mathcal{C}(X)$, then the algebra $\mathcal{A}$ has separation, non-vanishing property:

1. Suppose on the contrary that $\mathcal{A}$ is not separating, i.e., there exists $x_1, x_2 \in X$ such that $\phi(x_1) = \phi(x_2)$, $\forall \phi \in \mathcal{A}$.
   By the defintion of closure, it's clear that for given $S \subseteq (X,d)$, $\forall x \in \overline{S}$, there exists a sequence $\{S_n\}$ in S such that $S_n \to x$.
   Construct $f \in \mathcal{C}(X)$ defined by $f(x) = d(x, x_1)$. It follows that
   $$f(x_1) = 0, \quad f(x_2) = d(x_2, x_1) := k > 0$$
   Now we claim that $f \notin \overline{\mathcal{A}}$, since otherwise there exists $\{\phi_n\}$ in $\mathcal{A}$ such that $\phi_n \to f$, i.e.,
   $$\phi_n(x_1) \to f(x_1), \quad \phi_n(x_2) \to f(x_2), \quad \phi_n(x_1) = \phi_n(x_2), \forall n,$$
   i.e., $0 = f(x_1) = f(x_2) > 0$.
2. Suppose on the contrary that $\mathcal{A}$ is not non-vanishing, i.e., there exists some $x_0 \in X$ such that $\phi(x_0) = 0, \forall \phi \in \mathcal{A}$. Construct $g \in \mathcal{C}(X)$ defined by $g(x) = d(x, x_0) + 1$. Following the similar idea, we can show that there does not exist $\phi_n \in \mathcal{A}$ such that $\phi_n \to g$, i.e., $g \notin \overline{\mathcal{A}}$, which is a contradiction.

Example 5.4 1. Let $X \subseteq \mathbb{R}^n$ be a compact space. Then the polynomial ring

$$\mathbb{R}[x_1, \ldots, x_n] = \{\text{Polynomials in } n \text{ variables with coefficients in } \mathbb{R}\}$$

forms a dense set in $\mathcal{C}(X)$.

It's clear that the set $\mathbb{R}[x_1, \ldots, x_n]$ satisfies the separating and non-vanishing property.

For the special case $n=1$ and $X=[a,b]$, we get the Weierstrass Approximation Theorem.

2. In particular, when $X=S^1\subseteq\mathbb{R}^2$, we imply $\mathbb{R}[x,y]$ is dense in $\mathcal{C}(S^1)$.

5.2.1. Stone-Weierstrass Theorem in $\mathbb{C}$

Consider the circle $S^1\subseteq\mathbb{C}$ and the mappings

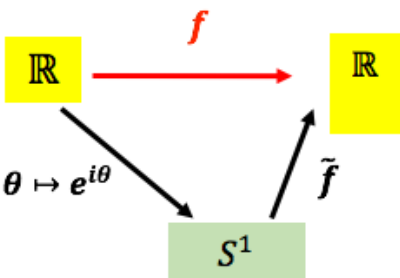
$$\begin{aligned} c: S^1 &\to \mathbb{R} & \qquad s: S^1 &\to \mathbb{R} \\ \text{with}\quad e^{i\theta} &\to \cos\theta & \qquad \text{with}\quad e^{i\theta} &\to \sin\theta \end{aligned}$$

are both continuous.

The algebra formed by s and c is given by

$$\mathcal{J} := \langle c,s\rangle = \operatorname{span}\{\cos^m\theta\sin^n\theta \mid m,n\in\mathbb{N}\}$$

1. The $\mathcal{J}$ satisfies both separating and non-vanishing property, which implies $\overline{\mathcal{J}}=\mathcal{C}(S^1)$.
2. Suppose $f:\mathbb{R}\to\mathbb{R}$ is a continuous, 2π-periodic mapping. It's easy to construct a continuous mapping $\tilde{f}:S^1\to\mathbb{R}$ such that the diagram below commutes:



Or equivalently, $f(\theta)=\tilde{f}(e^{i\theta})$ for some $\tilde{f}\in\mathcal{C}(S^1)$. Since $\overline{\mathcal{J}}=\mathcal{C}(S^1)$, we can approximate $\tilde{f}\in\mathcal{C}(S^1)$ by $\langle\cos\theta,\sin\theta\rangle$, which implies that the $f(\theta)$ can be approximated by

$$\sum_{m,n\in\mathbb{N}} a_{m,n}\cos^m\theta\sin^n\theta.$$

Since $\operatorname{span}\{\cos^m\theta\sin^n\theta\}_{m,n\in\mathbb{N}} = \operatorname{span}\{\cos(m\theta),\sin(n\theta),1\}_{m,n\in\mathbb{N}}$, we imply $f(\theta)$ can be approximated by

$$\sum_{m,n\in\mathbb{N}} a_m\cos(m\theta) + b_n\sin(n\theta).$$

Or equivalently, for any $\varepsilon > 0$, there exists $N > 0$ and $a_m, a_n \in \mathbb{R}$ such that

$$\left| f(\theta) - \left(a_0 + \sum_{m=1}^{N} a_m\cos(m\theta) + \sum_{n=1}^{N} b_n\sin(n\theta)\right)\right| < \varepsilon, \quad \forall\theta\in[0,2\pi]. \tag{5.1}$$

R The natural question is that do we have the following equation hold:

$$f(\theta) = a_0 + \sum_{m=1}^{\infty} a_m\cos(m\theta) + \sum_{n=1}^{\infty} b_n\sin(n\theta) \tag{5.2}$$

It seems that Eq.(5.2) above is equivalent to the expression in (5.1). However, unlike the Taylor expansion, the values of a_m, a_n, M, N may change once we switch the number $\varepsilon > 0$.

Therefore, Eq.(5.2) does not hold for most functions, but only for some functions with nice structure.

Fourier Analysis. Given the condition that the Eq.(5.2) holds. How can we get the values of a_m and b_n? The way is to take "inner product" between $f(\theta)$ and trigonometric functions. For example, by taking the inner product with $\cos(k\theta)$ for Eq.(5.2) both sides, we have

$$\begin{aligned}\int_{-\pi}^{\pi} f(\theta)\cos(k\theta)\,\mathrm{d}\theta &= \frac{a_0}{2}\int_{-\pi}^{\pi}\cos(k\theta)\,\mathrm{d}\theta \\ &\quad + \sum_{m=1}^{\infty} a_m\int_{-\pi}^{\pi}\cos(m\theta)\cos(k\theta)\,\mathrm{d}\theta + \sum_{m=1}^{\infty} b_n\int_{-\pi}^{\pi}\sin(n\theta)\cos(k\theta)\,\mathrm{d}\theta \\ &= \pi\cdot a_k\end{aligned}$$

Following the same trick, we obtain:

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(k\theta)\,\mathrm{d}\theta \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(k\theta)\,\mathrm{d}\theta \end{aligned} \tag{5.3}$$

Naturally, we define the fourier expansion for general $f(\theta)$, even though we don't verify whether (5.2) holds or not:

$$g_N(\theta) = \frac{a_0}{2} + \sum_{n=1}^{N} a_m \cos(m\theta) + \sum_{n=1}^{N} b_n \sin(n\theta),$$

where the term a_m and b_n follow the definition in (5.3). The natural question is that whether $g_N(\theta) \to f(\theta)$ as $N \to \infty$?

5.2.2. Baire Category Theorem

Motivation. The set $\mathcal{P}[a,b] \subseteq \mathcal{C}[a,b]$ is dense by Weierstrass Approximation. However, it is not "abundant" in $\mathcal{C}[a,b]$, just like $\mathbb{Q} \subseteq \mathbb{R}$ is dense in $\mathbb{R}$. (Every $r \in \mathbb{R}$ is a limit of a sequence in $\mathbb{Q}$)

The set $\mathbb{Q}$ is countable yet $\mathbb{R} \setminus \mathbb{Q}$ is uncountable, i.e., there are many more holes in $\mathbb{R} \setminus \mathbb{Q}$.

Definition 5.2 [Nowhere Dense] A subset $S \subseteq (X,d)$ is **nowhere dense** if $\overline{S}$ does not contain any open ball, i.e.,

$$X \setminus \overline{S} \text{ is dense in } X$$

For example, a single point is nowhere dense.

Theorem 5.1 Let $\{E_i\}_{i=1}^{\infty}$ be a collection of nowhere dense sets in a complete metric space (X,d). Then the set

$$\bigcup_{i=1}^{\infty} \overline{E_i}$$

also does not contain any open ball.

Proof. I have no time to review and modify the proof during the lecture. Therefore, we encourage the reader to go through the proof in the note

W,Ni & J. Wang (January, 2019). Lecture Notes for MAT2006. Retrieved from `https://walterbabyrudin.github.io/information/information.html`

Of course, I will also add the proof in this note during this week. ■