## 4.5. Wednesday for MAT3006

The quiz will be held on Wednesday.

**Reviewing.** Let's go through the proof for Weierstrass Theorem quickly.

- Study $Q_n(x) = c_n(1-x^2)^n$ and construct the approximate function

$$p_n(x) = \int_{-1}^{1} Q_n(t) f(x+t)\,\mathrm{d}t$$

- Show that

$$\begin{aligned}
|p_n(x) - f(x)| &\le \int_{-1}^{1} |f(x+t) - f(x)| Q_n(t)\,\mathrm{d}t \\
&= \left(\int_{\delta}^{1} + \int_{\delta}^{-\delta} + \int_{-1}^{-\delta}\right) |f(x+t) - f(x)| Q_n(t)\,\mathrm{d}t \\
&\le 4M\sqrt{n}(1-\delta^2)^n + \int_{\delta}^{-\delta} |f(x+t) - f(x)| Q_n(t)\,\mathrm{d}t \\
&\le 4M\sqrt{n}(1-\delta^2)^n + \varepsilon \cdot \int_{\delta}^{-\delta} Q_n(t)\,\mathrm{d}t \\
&\le 4M\sqrt{n}(1-\delta^2)^n + \varepsilon
\end{aligned}$$

  Therefore, $\|p_n - f\|_\infty \to 0$ as $n \to \infty$.

- Generalization for $\forall g \in \mathcal{C}[0,1]$: Recall that we have assumed $f(0) = f(1) = 0$. Now consider the general case, say

$$g(0) = a, \quad g(1) = b.$$

  Consider $f(x) := g(x) - l(x)$, where $l$ is the line segment from $(0,a)$ to $(1,b)$. Then we imply $|f(x) - p_n(x)| < \varepsilon$, i.e.,

$$|g(x) - (p_n(x) + l(x))| < \varepsilon, \quad \forall x.$$

- Generlization for $\forall h \in \mathcal{C}[a,b]$: Recall that we have restrict $f$ is continuous on $[0,1]$. For any $h \in \mathcal{C}[a,b]$, define $g(x) = h((b-a)x + a)$ for $x \in [0,1]$. Therefore,

$g \in \mathcal{C}[0,1]$, i.e., $|g(y) - p_n(y)| < \varepsilon, \forall y \in [0,1]$, which implies

$$|h((b-a)y+a) - p_n(y)| < \varepsilon, \quad \forall y \in [0,1]$$

Applying change of variables with $x = (b-a)y + a$, we imply

$$\left| h(x) - p_n\left(\frac{x-a}{b-a}\right) \right| < \varepsilon, \quad \forall x \in [a,b],$$

where $p_n(\cdot)$ is a polynomial function.

### 4.5.1. Stone-Weierstrass Theorem

The motivation is to generalize the Weierstrass approximation into the space $\mathcal{C}(X)$, where $(X,d)$ is a general compact space. Here $\mathcal{C}(X) := \{f : X \to \mathbb{R} \text{ is continuous}\}$. Note that

- $\mathcal{C}(X)$ has a norm:

  $$\|f\|_\infty := \sup\{f(x) \mid x \in X\}$$

  This is well-defined, since $f(X) \subseteq \mathbb{R}$ is compact, i.e., closed and bounded.

- $(\mathcal{C}(X), d_\infty)$ is complete. The proof follows similarly from the proof that $\mathcal{C}[a,b]$ is complete (see Example(**??**)).

R If $X$ is not compact, then the norm $\|\cdot\|_\infty$ is **not** well-defined on $\mathcal{C}(X)$, but this norm is still well-defined on the space

$$\mathcal{C}_b(X) = \{f : X \to \mathbb{R} \mid f \text{ is continous and bounded}\}.$$

If $X$ is compact, then $\mathcal{C}(X) = \mathcal{C}_b(X)$.

**Definition 4.10** [Separation Property] Let $(X,d)$ be any metric space, and $\mathcal{A} \subseteq \mathcal{C}_b(X)$ is an algebra (closed under linear combination and pointwise product), then

1. $\mathcal{A}$ is said to be equipped with the **separation property** if for any $x_1 \neq x_2 \in X$, there exists $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$
2. $\mathcal{A}$ is said to be equipped with the **nonvanishing property** if for any $x \in X$, there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$.

■ **Example 4.4** Suppose that $X := S^1 := \{e^{i\theta} \mid \theta \in [0,2\pi]\} \subseteq \mathbb{C} \cong \mathbb{R}^2$, and consider the algebra

$$\mathcal{A} = \langle g \rangle := \operatorname{span}\{1, g, g^2, \dots\}$$

Define $g : S^1 \to \mathbb{R}$ as $g(e^{i\theta}) = \cos\theta$. Note that

1. $\mathcal{A}$ does not satisfy the separation property: take $e^{i\theta}, e^{i(2\pi-\theta)}$
2. However, $\mathcal{A}$ satisfies the nonvanishing property. Consider the special element of $\mathcal{A}$: $f \equiv 1$.

**Theorem 4.4** — **Stone-Weierstrass Theorem.** Let $(X,d)$ be a compact space, and $\mathcal{A} \subseteq \mathcal{C}(X)$ is an algebra. Then $\overline{A} = \mathcal{C}(X)$ iff $A$ satisfies both the **nonvanishing** and **separation** property.

Before going through the proof, we establish two lemmas below:

**Proposition 4.12** If both $f, g$ belong to the algebra $\mathcal{A}$, then $\max\{f,g\} \in \mathcal{A}$ and $\min\{f,g\} \in \overline{\mathcal{A}}$.

*Proof.* Since

$$\begin{aligned}
\max\{f,g\} &= \frac{1}{2}(f+g) + \frac{1}{2}|f-g| \\
\min\{f,g\} &= \frac{1}{2}(f+g) - \frac{1}{2}|f-g|,
\end{aligned}$$

it suffices to show $|h| \in \overline{\mathcal{A}}$ given that $h \in \mathcal{A}$.

Let $M = \max\{|h(x)| \mid x \in X\}$. Consider the function (w.r.t. $t$) $|t| \in \mathcal{C}[-M, M]$. By Weierstrass approximation, there exists a polynomial $p$ such that $||t| - p(t)| < \varepsilon$, which implies

$$||h(x)| - p(h(t))| < \varepsilon.$$

Note that $p(h(t))$ is a polynomial of $h(t)$, and therefore an element from the algebra $\mathcal{A}$. Therefore, $|h|$ can be approximated by some element from $\mathcal{A}$, i.e., $|h| \in \overline{\mathcal{A}}$. ■

**Proposition 4.13** Let $\mathcal{A} \subseteq \mathcal{C}(X)$ be an algebra satisfying the separation property and non-vanishing property. Then for all $x_1 \neq x_2 \in X$, and any $\alpha, \beta \in \mathbb{R}$, there exists $f \in \mathcal{A}$ such that

$$\begin{cases} f(x_1) = \alpha \\ f(x_2) = \beta \end{cases}$$

*Proof.* By separation property, there exists $h \in \mathcal{A}$ such that $h(x_1) \neq h(x_2)$.

1. We claim that we can construct a new $h$ such that

$$h(x_1) \neq h(x_2), \quad h(x_1) \neq 0, \quad h(x_2) \neq 0 \tag{4.5}$$

   (a) If both $h(x_1), h(x_2) \neq 0$, we have done.
   (b) If not, suppose $h(x_1) = 0$. By non-vanishing property, there eixsts $p \in \mathcal{A}$ such that $p(x_1) \neq 0$. Then some linear transformation of $h$ and $p$ will do the trick. (hint: construct $t$ such that $h \leftarrow h + t \cdot p$ gives the desired result.)

2. Now suppose the requirement (4.5) is met. Consider the function

$$f(x) = ah(x) + bh^2(x) \in \mathcal{A},$$

   where $a, b$ are two parameters to be determined.

   Indeed, it suffices to find $a, b$ such that $f(x_1) = \alpha, f(x_2) = \beta$, or equivalently, solve the linear system

$$f(x_1) = ah(x_1) + bh^2(x_1) = \alpha$$
$$f(x_2) = ah(x_2) + bh^2(x_2) = \beta$$

Since the determinant of the linear system is not equal to 0, $a, b$ can be clearly found.

The proof is complete. ■

*Necessity part of the proof.* Given that $\mathcal{A}$ has separation and non-vanishing, we aim to show $\overline{\mathcal{A}} = \mathcal{C}(X)$.

1. Take any $f \in \mathcal{C}(X)$. By proposition (4.13), for any $x, y \in X$, there exists $\phi_{x,y} \in \mathcal{A}$ such that
$$\begin{cases} \phi_{x,y}(x) = f(x) \\ \phi_{x,y}(y) = f(y) \end{cases}.$$
Construct the open set $U_{x,y} = (f - \phi_{x,y})^{-1}((-\varepsilon, \varepsilon))$, i.e.,
$$U_{x,y} = \{t \in X \mid \phi_{x,y}(t) - \varepsilon < f(t) < \phi_{x,y}(t) + \varepsilon\}.$$

2. It's clear that $x, y \in U_{x,y}$. For fixed $y \in X$, the collection $\{U_{x,y}\}_{x \in X}$ forms an open cover of $X$. By the compactness of $X$, there exists the finite subcover
$$\{U_{x_1,y}, \ldots, U_{x_N,y}\} \supseteq X.$$
By proposition (4.12), the function $\phi_y := \max\{\phi_{x_1,y}, \ldots, \phi_{x_N,y}\} \in \overline{\mathcal{A}}$. Furthermore, for $\forall x \in X$, we imply there exists some $U_{x_i,y} \ni x$, i.e.,
$$f(x) < \phi_{x_i,y}(x) + \varepsilon \implies f(x) < \phi_y(x) + \varepsilon, \ \forall x \in X.$$

3. Also, consider $V_y = \cap_{i=1}^N U_{x_i,y}$, which is the open set containing $y$, and $\{V_y\}_{y \in X}$

covers $X$ (why?). Note that for any $x \in V_y$, we imply $x \in U_{x_i,y}, \forall i$, i.e.,

$$\phi_{x_i,y}(x) - \varepsilon < f(x), \quad \forall i \implies \phi_y(x) - \varepsilon < f(x),\ \forall x \in V_y.$$

By the compactness of $X$ again, we take finite subcover $\{V_{y_j}\}_{j=1}^{M}$ and define

$$\phi(x) := \min\{\phi_{y_1}(x), \dots, \phi_{y_M}(x)\} \in \overline{\mathcal{A}}.$$

Therefore, for any $x \in X$ we imply $x \in V_{y_m}$, i.e.,

$$\phi_{y_m}(x) - \varepsilon < f(x) \implies \phi(x) - \varepsilon < f(x) \tag{4.6}$$

4. Also, from (2) we have obtained $f(x) < \phi_y(x) + \varepsilon$ for $\forall y \in X$. In particular,

$$f(x) < \phi_{y_m}(x) + \varepsilon,\ \forall m = 1, \dots, M \tag{4.7}$$

   Combining (4.6) and (4.7), we imply $|\phi(x) - f(x)| < \varepsilon$.

Therefore, we have constructed a function $\phi \in \overline{\mathcal{A}}$ such that $|\phi(x) - f(x)| < \varepsilon$, which implies $f \in \overline{\overline{\mathcal{A}}} = \overline{\mathcal{A}}$. The proof is complete. ∎