4.2. Monday for MAT3006

Our first quiz will be held on next Wednesday.

Reviewing.

- Picard Lindelof Theorem on ODEs. e.g., consider

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x}{1-y}, \ (x,y) \in G := (-\infty,\infty) \times (-\infty,1) \\ y(0) = 2 \end{cases}$$

Since $f \in \mathcal{C}^1(G)$ satisfies the Lipschitz condition on some closed ball of the point (x_0, y_0), the setting for Picard Lindelof Theorem is satisfied, and the solution is uniquely given by:

$$y = 1 + \sqrt{1-x^2}, \ -1 < x < 1.$$

Therefore, the maximal interval of existence is given by $(-1,1)$. In order to restrict G to be open to construct a closed ball of (x_0, y_0), we need the initial condition $y(0) \neq 1$.

4.2.1. Generalization into System of ODEs

Formal Setting of System of ODEs. Consider the system of ODEs

$$\begin{cases} y_1'(x) = f_1(x, y_1(x), \dots, y_n(x)) \\ \quad \vdots \\ y_n'(x) = f_n(x, y_1(x), \dots, y_n(x)) \end{cases} \qquad \begin{cases} y_1(\alpha) = \beta_1 \\ \quad \vdots \\ y_n(x) = \beta_n \end{cases}$$

It's convenient to denote

$$\boldsymbol{y}(x) = \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix} \in \mathcal{C}(\mathbb{R}, \mathbb{R}^n), \quad \boldsymbol{f}(x, \boldsymbol{y}) = \begin{pmatrix} f_1(x, \boldsymbol{y}) \\ \vdots \\ f_n(x, \boldsymbol{y}) \end{pmatrix}, \quad \boldsymbol{\beta} := \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

Here the notation $\mathcal{C}(X,Y)$ denotes the set of bounded continuous mapping from X to Y. Therefore we can express the system of ODE as a compact form:

$$\begin{cases} \boldsymbol{y}' = \boldsymbol{f}(x,\boldsymbol{y}) \\ \boldsymbol{y}(\alpha) = \boldsymbol{\beta} \end{cases}$$

Generalization of Picard Lindelof Theorem. Consider the rectangle

$$S = \{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}^n \mid \alpha - a \le x \le \alpha + a, \beta_i - b_i \le y_i \le \beta_i + b_i, i = 1,\dots,n\}$$

Suppose that

- $\|\boldsymbol{f}(x,\boldsymbol{y})\| \le M, \forall (x,\boldsymbol{y}) \in S$

- $\|\boldsymbol{f}(x,\boldsymbol{y}) - \boldsymbol{f}(x,\boldsymbol{y}')\| \le L \cdot \|\boldsymbol{y} - \boldsymbol{y}'\|$ for $\forall x \in [\alpha - a, \alpha + a]$

Then consider the complete metric space

$$X = \{\boldsymbol{y} \in \mathcal{C}([\alpha - a, \alpha + a], \mathbb{R}^n) \mid \beta_i - b_i \le y_i(x) \le \beta_i + b_i\}$$

(Verification of completeness: if Y is complete, then $\mathcal{C}(X,Y)$ is complete.) Under this setting, the similar argument gives the Picard-Lindelof for system of ODEs.

Higher Order ODEs. Note that there is a standard way to transform the ODE with higher order derivatives into a system of first order ODEs. Suppose we want to solve the initival value problem

$$\begin{cases} y^{(m)} = f(x,y,y',\dots,y^{(m-1)}) \\ y(\alpha) = \beta_0,\ y'(\alpha) = \beta_1,\dots,y^{(m-1)}(\alpha) = \beta_{m-1} \end{cases}$$

We can define the variables

$$\begin{pmatrix} y_{m-1}(x) \\ \vdots \\ y_1(x) \\ y_0(x) \end{pmatrix} = \begin{pmatrix} y^{(m-1)}(x) \\ \vdots \\ y'(x) \\ y(x) \end{pmatrix}$$

which gives an equivalent system of ODE:

$$\begin{cases} y'_{m-1} = f(x, y_0, \dots, y_{m-1}) \\ y'_{m-2} = y_{m-1} \\ \quad\vdots \\ y'_0 = y_1 \end{cases}, \text{ with } \begin{cases} y_{m-1}(\alpha) = \beta_{m-1} \\ y_{m-2}(\alpha) = \beta_{m-2} \\ \quad\vdots \\ y_0(\alpha) = \beta_0 \end{cases}$$

4.2.2. Stone-Weierstrass Theorem

Under the compact metric space X, the goal is to approximate **any** functions in $\mathcal{C}(X)$. For example, under $X = [a, b]$, one can apply Taylor polynomials $p_n(x)$ to approximate differentiable functions:

$$\|f(x) - p_n(x)\|_\infty < \varepsilon, \text{ for large } n.$$

To formally describe the phenomenon for the approximation of **any** functions in $\mathcal{C}(X)$, we need to describe the set of approximate functions, which usually obtains a common property:

Definition 4.3 [Algebra] A subset $\mathcal{A} \subseteq \mathcal{C}(X)$ (where X is a general space) is an **algebra** if the following holds:

- If $f_1, f_2 \in \mathcal{A}$, then $\alpha f_1 + \beta f_2 \in \mathcal{A}$
- If $f_1, f_2 \in \mathcal{A}$, then $f_1 \cdot f_2 \in \mathcal{A}$

■ **Example 4.1** 1. $\mathcal{A} = \mathcal{C}(X)$ is an algebra.

2. $X = [a,b]$, then $\mathcal{A} = P[a,b] = \{\text{All polynomials } p(x)\}$ is an algebra. ■

The goal is to approximate any $f \in \mathcal{C}(X)$ by $p \in \mathcal{A}$, i.e., for $\forall f \in \mathcal{C}(X)$, there exists $p \in \mathcal{A}$ such that

$$\|f - p\|_\infty < \varepsilon, \ \forall \varepsilon > 0.$$

In other words, we aim to find an algebra $\mathcal{A} \subseteq \mathcal{C}(X)$ such that $\overline{\mathcal{A}} = \mathcal{C}(X)$, i.e., $\mathcal{A}$ is dense in $\mathcal{C}(X)$.

Theorem 4.2 — **Weierstrass Approximation.** $\mathcal{P}[a,b]$ is dense in $\mathcal{C}[a,b]$.

Proof. Consider any function $f \in \mathcal{C}[0,1]$. By rescaling, assume that $f \in \mathcal{C}[0,1]$. By subtracting a linear function $\ell(x)$, assume that $f(0) = f(1) = 0$. Then we extend $f(x)$ into $\mathbb{R}$ by setting $f(x) = 0, \forall x \notin [0,1]$.

- **Step 1: Construction of approximate function**: Consider the **Landaus kernel function**

$$Q_n(x) = \begin{cases} c_n \cdot (1-x^2)^n, & -1 \le x \le 1 \\ 0, & |x| > 1 \end{cases}$$

  where c_n is chosen such that $\int Q_n(x)\,\mathrm{d}x = 1$. Then construct the approximation of f by defining

$$p_n(x) := Q_n * f = \int_{-1}^{1} f(x+t) Q_n(t)\,\mathrm{d}t$$

  The intuition behind this construction is that as $n \to \infty$, $Q_n(x) \to \delta(x)$, where

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \implies \int_{-1}^{1} f(x+t)\delta(t)\,\mathrm{d}t = f(x).$$

Step 2: Argue that $p_n(x) \in \mathcal{P}[a,b]$: Now it's clear that

$$
\begin{aligned}
p_n(x) &= \int_{-1}^{1} f(x+t)Q_n(t)\,\mathrm{d}t &\text{(4.2a)}\\
&= \int_{-x}^{1-x} f(x+t)Q_n(t)\,\mathrm{d}t &\text{(4.2b)}\\
&= \int_{-1}^{1} f(u)\cdot Q_n(u-x)\,\mathrm{d}u &\text{(4.2c)}\\
&= \int_{-1}^{1} f(u)\cdot (1-(u-x)^2)^n\,\mathrm{d}u, &\text{(4.2d)}
\end{aligned}
$$

where (4.2b) is because that $f = 0$, for $x \notin [0,1]$ and $Q_n = 0$ for $|x| > 1$; (4.2c) is by change of variables; and (4.2d) is by substitution of $Q_n(x)$. Therefore, p_n is still a polynomial of x.

- **Step 3: Construct an upper bound on c_n:** It's clear that

$$
\begin{aligned}
c_n^{-1} &= \int_{-1}^{1} (1-x^2)^n\,\mathrm{d}x\\
&= 2\int_{0}^{1} (1-x^2)^n\,\mathrm{d}x\\
&\ge 2\int_{0}^{1} (1-nx^2)\,\mathrm{d}x\\
&\ge 2\int_{0}^{1/\sqrt{n}} (1-nx^2)\,\mathrm{d}x\\
&= 2\left(\frac{1}{\sqrt{n}} - \frac{1}{3\sqrt{n}}\right) > \frac{1}{\sqrt{n}}
\end{aligned}
$$

and therefore $c_n < \sqrt{n}$. As a result, for any fixed $\delta \in (0,1)$, we imply

$$
Q_n(x) \le \sqrt{n}(1-\delta^2)^n, \qquad \forall x \in [\delta, 1],
$$

which implies $Q_n(x) \to 0$ uniformly on $[\delta, 1]$.

- **Step 4: Show that $\|p_n - f\|_\infty \to 0$.** Since f is continuous, for given $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$
|f(x) - f(y)| < \varepsilon, \quad \text{when } |x-y| < \delta, x, y \in [0,1].
$$

Therefore, for any $x \in [0,1]$, and for sufficiently large n,

$$\begin{aligned}
|p_n(x) - f(x)| &= \left| \int_{-1}^{1} f(x+t)Q_n(t) - \int_{-1}^{1} f(x)Q_n(t)\,\mathrm{d}t \right| && (4.3a)\\
&\leq \int_{-1}^{1} |f(x+t) - f(x)| Q_n(t)\,\mathrm{d}t && (4.3b)\\
&\leq 2M \int_{-1}^{-\delta} Q_n(t)\,\mathrm{d}t + \frac{\varepsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\,\mathrm{d}t + 2M\int_{\delta}^{1} Q_n(t)\,\mathrm{d}t && (4.3c)\\
&\leq 4M\sqrt{n}(1-\delta^2)^n + \frac{\varepsilon}{2} && (4.3d)\\
&\leq \varepsilon && (4.3e)
\end{aligned}$$

where (4.3c) is by separating the integrand into three parts, and then upper bounding $|f(x+t) - f(x)|$ by $2M := 2\sup_x |f(x)|$ for the integrand $t \in [-1, \delta) \cup (\delta, 1]$, and upper bounding $|f(x+t) - f(x)|$ by $\frac{\varepsilon}{2}$ due to the continuity of f for the integrand $t \in [\delta, \delta]$; (4.3e) is by choosing n sufficiently enough to make $4M\sqrt{n}(1-\delta^2)^n$ sufficiently small.

Therefore $\|p_n - f\|_\infty = \max_{x\in[0,1]} |p_n(x) - f(x)| < \varepsilon$ for large n. The proof is complete.

■