3.5. Wednesday for MAT3006

3.5.1. Remarks on Contraction

Reviewing.

- Suppose $E \subseteq X$ with X being complete, then E is closed in X iff E is complete
- Suppose $E \subseteq X$, then E is closed in X if E is complete.
- Contraction Mapping Theorem
- Classification for the Convergence of Newton's method: the Newton's method aims to find the fixed point of T.

  $$T : \mathbb{R} \to \mathbb{R}, \quad T(x) = x - \frac{f(x)}{f''(x)}$$

  In the last lecture we claim that there exists $[r-\varepsilon, r+\varepsilon]$ such that $\sup_{[r-\varepsilon,r+\varepsilon]} |T'(x)| < 1$.

  Note that we doesn't make our statement rigorous enough. we need to furthermore show that $T(X) \subseteq X$:

  - $T : [r-\varepsilon, r+\varepsilon] \to [r-\varepsilon, r+\varepsilon]$, since

    $$|T(x) - r| = |T(x) - T(r)| = |T'(s)||x-r| \le \sup_{[r-\varepsilon,r+\varepsilon]} |T'(s)||x-r| < |x-r|$$

    Therefore, if $x \in [r-\varepsilon, r+\varepsilon]$, then $T(x) \in [r-\varepsilon, r+\varepsilon]$.
  - T is a contraction:

    $$|T(x) - T(y)| < \tau \cdot |x-y|$$

  Therefore, applying contraction mapping theorem gives the desired result.

3.5.2. Picard-Lindelof Theorem

Consider solving the the initival value problem given below

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \\ y(\alpha) = \beta \end{cases} \implies y(x) = \beta + \int_{\alpha}^{x} f(t,y(t))\,\mathrm{d}t \tag{3.11}$$

Definition 3.5 Let $R = [\alpha - a, \alpha + a] \times [\beta - b, \beta + b]$. Then the function $f(x,y)$ satisfies the **Lipschitz condition** on R if there exists $L > 0$ such that

$$|f(x,y_1) - f(x,y_2)| < L \cdot |y_1 - y_2|, \quad \forall (x,y_i) \in R \tag{3.12}$$

The smallest number $L^* = \inf\{L \mid \text{The relation (3.12) holds for } L\}$ is called the **Lipschitz constant** for f.

■ **Example 3.11** A $\mathcal{C}^1$-function f(x,y) in a rectangle automatically satisfies the Lipschitz condition:

$$f(x,y_1) - f(x,y_2) \overset{\text{Appling MVT}}{=} \frac{\partial f}{\partial y}(x,\tilde{y})(y_1 - y_2)$$

Since $\frac{\partial f}{\partial y}$ is continuous on R and thus bounded, we imply

$$|f(x,y_1) - f(x,y_2)| < L \cdot |y_1 - y_2|, \quad \forall (x,y_i) \in R$$

where

$$L = \max\left\{ \text{abs}\left(\frac{\partial f}{\partial y}\right) \middle| (x,y) \in R \right\}$$

Theorem 3.4 — Picard-Lindelof Theorem (existence part). Suppose $f \in \mathcal{C}(R)$ be such that f satisfies the Lipschitz condition, then there exists $a'' \in (0,a]$ such that (**??**) is solvable with $y(x) \in \mathcal{C}([\alpha - a'', \alpha + a''])$.

Proof. Consider the complete metric space

$$X = \{y(x) \in \mathcal{C}([\alpha - a, \alpha + a]) \mid \beta - b \leq y(x) \leq \beta + b\},$$

with the mapping $T : X \to X$ defined as

$$(Ty)(x) = \beta + \int_{\alpha}^{x} f(t, y(t))\,\mathrm{d}t$$

It suffices to show that T is a contraction, but here we need to estrict a a smaller number as follows:

1. Well-definedness of T: Take $M := \sup\{f(x,y) \mid (x,y) \in R\}$ and construct $a' = \min\{b/M, a\}$. Consider the complete matric space

   $$X = \{y(x) \in \mathcal{C}([\alpha - a', \alpha + a']) \mid \beta - b \leq y(x) \leq \beta + b\}$$

   which implies that

   $$|(Ty)(x) - \beta| \leq \left| \int_{\alpha}^{x} f(t, y(t))\,\mathrm{d}t \right| \leq M|x - \alpha| \leq Ma' \leq b,$$

   i.e., $T(X) \subseteq X$, and therefore $T : X \to X$ is well-defined.
2. Contraction for T: Construct $a'' \in \min\{a', \frac{1}{2L^*}\}$, where L^* is the Lipschitz constant for f. and consider the complete metric space

   $$X = \{y(x) \in \mathcal{C}([\alpha - a'', \alpha + a'']) \mid \beta - b \leq y(x) \leq \beta + b\}$$

   Therefore for $\forall x \in [\alpha - a'', \alpha + a'']$ and the mapping $T : X \to X$,

   $$\begin{aligned} |[T(y_1) - T(y_2)](x)| &\leq \left| \int_{\alpha}^{x} [f(t, y_2(t)) - f(t, y_1(t))]\,\mathrm{d}t \right| \\ &\leq \int_{\alpha}^{x} |f(t, y_2) - f(t, y_1)|\,\mathrm{d}t \leq \int_{\alpha}^{x} L^*|y_2(t) - y_1(t)|\,\mathrm{d}t \\ &\leq L^*|x - \alpha| \sup |y_2(t) - y_1(t)| \leq L^* a'' d_\infty(y_2, y_1) \leq \frac{1}{2} d_\infty(y_2, y_1) \end{aligned}$$

Therefore, we imply $d_\infty(Ty_2, Ty_1) \leq \frac{1}{2} d_\infty(y_2, y_1)$, i.e., T is a contraction.

Applying contraction mapping theorem, there exists $y(x) \in X$ such that $Ty = y$, i.e.,

$$y = \beta + \int_\alpha^x f(t, y(t))\,\mathrm{d}t$$

Thus y is a solution for the IVP (3.11). ■

R On $[\alpha - a'', \alpha + a'']$, we can solve the IVP (3.11) by recursively applying T:

$$\begin{aligned}
y_0(x) &= \beta, \qquad \forall x \in [\alpha - a'', \alpha + a''] \\
y_1 &= T(y_0) = \beta + \int_\alpha^x f(t, \beta)\,\mathrm{d}t \\
y_2 &= T(y_1) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

By studying (3.11) on different rectangles, we are able to show the uniqueness of our solution:

Proposition 3.8 Suppose f satisfies the Lipschitz conditon, and y_1, y_2 are two solutions of (3.11), where y_1 is defined on $x \in I_1$, and y_2 is defined on $x \in I_2$. Suppose $I_1 \cap I_2 \neq \emptyset$ and y_1, y_2 share the same initial value condition $y(\alpha) = \beta$. Then $y_1(x) = y_2(x)$ on $I_1 \cap I_2$.

Proof. Suppose $I_1 \cap I_2 = [p, q]$ and let $z := \sup\{x \mid y_1 \equiv y_2 \text{ on } [\alpha, x]\}$. It suffices to show $z = q$. Now suppose on the contrary that $z < q$, and consider the subtraction $|y_1 - y_2|$:

$$y_i = \beta + \int_\alpha^x f(t, y_i)\,\mathrm{d}t \implies |y_1 - y_2| = \left| \int_z^x f(t, y_1) - f(t, y_2)\,\mathrm{d}t \right|.$$

Construct an interval $I^* = [z - \frac{1}{2L^*}, z + \frac{1}{2L^*}] \cap [p, q]$, and let $x_m = \arg\max_{x \in I^*} |y_1(x) -$

$y_2(x)|$, which implies for $\forall x \in I^*$,

$$\begin{aligned}
|y_1(x) - y_2(x)| &= \left| \int_z^x f(t,y_1) - f(t,y_2)\,\mathrm{d}t \right| \\
&\le \int_z^x |f(t,y_1(t)) - f(t,y_2(t))|\,\mathrm{d}t \\
&\le L^* \int_z^x |y_1(x) - y_2(x)|\,\mathrm{d}t \\
&\le L^* |x-z| |y_1(x_m) - y_2(x_m)| \\
&\le \frac{1}{2} |y_1(x_m) - y_2(x_m)|.
\end{aligned}$$

Taking $x = x_m$, we imply $y_1 \equiv y_2$ for $\forall x \in I^*$, which contradicts the maximality of z. ■

Combining Theorem (3.4) and proposition (3.8), we imply the existence of a unique "maximal" solution for the IVP (3.11), i.e., the unique solution is defined on a maximal interval.

Corollary 3.3 Let $U \subseteq \mathbb{R}^2$ be an open set such that $f(x,y)$ satisfies the Lipschitz condition for any $[a,b] \times [c,d] \subseteq U$, then there exists x_m and x_M in $\overline{\mathbb{R}}$ such that

- The IVP (3.11) admits a solution $y(x)$ for $x \in (x_m, x_M)$, and if y^* is another solution of (3.11) on some interval $I \subseteq (x_m, x_M)$, then $y \equiv y^*$ on I.
- Therefore $y(x)$ is maximally defined; and $y(x)$ is unique.

▪ **Example 3.12** Consider the IVP

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2 y^{1/5} \\ y(0) = C \end{cases} \implies \frac{\partial f}{\partial y} = \frac{x^2}{5y^{4/5}}.$$

- Taking $U = \mathbb{R} \times (0,\infty)$ implies $y = \left(\frac{4x^3}{15} + c^{4/5}\right)^{5/4}$, defined on $(\sqrt[3]{-15/4c^{4/5}}, \infty)$.
- When $c = 0$, $f(x,y)$ does not satisfy the Lipschitz condition. The uniqueness of solution does not hold.

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