3.2. Monday for MAT3006

Reviewing.

1. Compactness/Sequential Compactness:
   - Equivalence for metric space
   - Stronger than closed and bounded
2. Completeness:
   - The metric space (E,d) is complete if every Cauchy sequence on E is convergent.
   - $\mathbb{P}[a,b] \subseteq \mathcal{C}[a,b]$ is not complete:
   $$f_N(x) = \sum_{n=0}^{N} (-1)^n \frac{x^{2n}}{(2n)!} \to \cos x,$$
   while $\cos x \notin \mathcal{P}[a,b]$.

3.2.1. Remarks on Completeness

Proposition 3.3 Let (X,d) be a metric space.

1. If X is complete and $E \subseteq X$ is closed, then E is complete.
2. If $E \subseteq X$ is complete, then E is closed in X.
3. If $E \subseteq X$ is compact, then E is complete.

Proof.
1. Every Cauchy sequence $\{e_n\}$ in $E \subseteq X$ is also a Cauchy sequence in E. Therefore we imply $\{e_n\} \to x \in X$, due to the completeness of X. Due to the closedness of E, the limit $x \in E$, i.e., E is complete.
2. Consider any convergent sequence $\{e_n\}$ in E, with some limit $x \in X$. We imply $\{e_n\}$ is Cauchy and thus $\{e_n\} \to e \in E$, due to the completeness of E. By the uniqueness of limits, we must have $x = z \in E$, i.e., E is closed.
3. Consider a Cauchy sequence $\{e_n\}$ in E. There exists a subsequence $\{e_{n_j}\} \to e \in E$, due to the sequential compactness of E.

It follows that for large n and j,

$$d(e_n, e) \overset{\text{(a)}}{\leq} d(e_n, e_{n_j}) + d(e_{n_j}, e) \overset{\text{(b)}}{<} \varepsilon$$

where (a) is due to triangle inequality and (b) is due to the Cauchy property of $\{e_n\}$ and the convergence of $\{e_{n_j}\}$.

Therefore, we imply $\{e_n\} \to e \in E$, i.e., E is complete.

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R Given any metric space that may not be necessarily complete, we can make the union of it with another space to make it complete, e.g., just like the completion from $\mathbb{Q}$ to $\mathbb{R}$.

3.2.2. Contraction Mapping Theorem

The motivation of the contraction mapping theorem comes from solving an equation $f(x)$. More precisely, such a problem can be turned into a problem for fixed points, i.e., it suffices to find the fixed points for $g(x)$, with $g(x) = f(x) + x$.

Definition 3.3 Let (X, d) be a metric space. A map $T : (X, d) \to (X, d)$ is a **contraction** if there exists a constant $\tau \in (0, 1)$ such that

$$d(T(x), T(y)) < \tau \cdot d(x, y), \quad \forall x, y \in X$$

A point x is called a fixed point of T if $T(x) = x$.

R All contractions are continuous: Given any convergence sequence $\{x_n\} \to x$, for $\varepsilon > 0$, take N such that $d(x_n, x) < \frac{\varepsilon}{\tau}$ for $n \geq N$. It suffices to show the convergence of $\{T(x_n)\}$:

$$d(T(x_n), T(x)) < \tau \cdot T(x_n, x) < \tau \cdot \frac{\varepsilon}{\tau} = \varepsilon.$$

Therefore, the contraction is Lipschitz continuous with Lipschitz constant τ.

Theorem 3.2 — Contraction Mapping Theorem / Banach Fixed Point Theorem. Every contraction T in a **complete** metric space X has a unique fixed point.

■ **Example 3.5**
1. The mapping $f(x) = x + 1$ is not a contraction in $X = \mathbb{R}$, and it has no fixed point.
2. Consider an in-complete space $X = (0,1)$ and a contraction $f(x) = \frac{x+1}{2}$. It doesn't admit a fixed point on X as well.

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Proof. Pick any $x_0 \in X$, and define a sequence recursively by setting $x_{n+1} = T(x_n)$ for $n \geq 0$.

1. Firstly show that the sequence $\{x_n\}$ is Cauchy.

   We can upper bound the term $d(T^n(x_0), T^{n-1}(x_0))$:

$$d(T^n(x_0), T^{n-1}(x_0)) \leq \tau d(T^{n-1}(x_0), T^{n-2}(x_0)) \leq \cdots \leq \tau^{n-1} d(T(x_0), x_0) \tag{3.4}$$

   Therefore for any $n \geq m$, where m is going to be specified later,

$$\begin{aligned} d(x_n, x_m) &= d(T^n(x_0), T^m(x_0)) && \text{(3.5a)} \\ &\leq \tau d(T^{n-1}(x_0), T^{m-1}(x_0)) \leq \cdots \leq \tau^m d(T^{n-m}(x_0), x_0) && \text{(3.5b)} \\ &\leq \tau^m \sum_{j=1}^{n-m} \tau^{n-m-j} d(T(x_0), x_0) && \text{(3.5c)} \\ &< \frac{\tau^m}{1-\tau} d(T(x_0), x_0) && \text{(3.5d)} \\ &\leq \varepsilon && \text{(3.5e)} \end{aligned}$$

   where (3.5b) is by repeatedly applying contraction property of d; (3.5c) is by applying the triangle inequality and (3.4); (3.5e) is by choosing sufficiently large m such that $\frac{\tau^m}{1-\tau} d(T(x_0), x_0) < \varepsilon$.

   Therefore, $\{x_n\}$ is Cauchy. By the completeness of X, we imply $\{x_n\} \to x \in X$.

2. Therefore, we imply

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = T(x),$$

i.e., x is a fixed point of T.

Now we show the uniqueness of the fixed point. Suppose that there is another fixed point $y \in X$, then

$$d(x,y) = d(T(x),T(y)) < \tau \cdot d(x,y) \implies d(x,y) < \tau d(x,y), \quad \tau \in (0,1),$$

and we conclude that $d(x,y) = 0$, i.e., $x = y$.

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■ **Example 3.6** [Convergence of Newton's Method] The Newton's method aims to find the root of $f(x)$ by applying the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x)}$$

Suppose r is a root for f, the pre-assumption for the convergence of Newton's method is:

1. $f'(r) \neq 0$
2. $f \in \mathcal{C}^2$ on some neighborhood of r

Proof. 1. We first show that there exists $[r-\varepsilon, r+\varepsilon]$ such that the mapping

$$T : \mathcal{C}[r-\varepsilon, r+\varepsilon] \to \mathbb{R}, \quad f(x) \mapsto x - \tfrac{f(x)}{f'(x)}$$

satisfies $|T'(x)| < 1$ for $\forall x \in [r-\varepsilon, r+\varepsilon]$:

Note that $T'(x) = \frac{f(x)}{[f'(x)]^2} f''(x)$, and we define $h(x) = |T'(x)|$.

It's clear that $h(r) = 0$ and $h(x)$ is continuous, which implies

$$r \in h^{-1}((-1,1)) \implies B_\rho(r) \subseteq h^{-1}((-1,1)) \text{ for some } \rho > 0.$$

Or equivalently, $h((r-\rho,r+\rho)) \subseteq (-1,1)$. Take $\varepsilon = \frac{\rho}{2}$, and the result is obvious.

2. Therefore, any $x,y \in [r-\varepsilon, r+\varepsilon]$,

$$\begin{aligned} d(T(x),T(y)) &:= |T(x) - T(y)| && \text{(3.6a)} \\ &= |T'(\xi)||x-y| && \text{(3.6b)} \\ &\leq \max_{\xi \in [r-\varepsilon, r+\varepsilon]} |T'(\xi)||x-y| && \text{(3.6c)} \\ &< m \cdot |x-y| && \text{(3.6d)} \end{aligned}$$

where (3.6b) is by applying MVT, and ξ is some point in $[r-\varepsilon, r+\varepsilon]$; we assume that $\max_{\xi \in [r-\varepsilon, r+\varepsilon]} |T'(\xi)| < m$ for some $m < 1$ in (3.6d).

Therefore, $T \in \mathcal{C}[r-\varepsilon, r+\varepsilon]$ is a contraction. By applying the contraction mapping theorem, there exists a unique fixed point near $[r-\varepsilon, r+\varepsilon]$:

$$x - \frac{f(x)}{f'(x)} = x \implies \frac{f(x)}{f'(x)} = 0 \implies f(x) = 0,$$

i.e., we obtain a root $x = r$. ■

Summary: if we use Newton's method on any point between $[r-\varepsilon, r+\varepsilon]$ where $f(r) = 0$ and ε is sufficiently small, then we will eventually get close to r. ▪

3.2.3. Picard Lindelof Theorem

We will use Banach fixed point theorem to show the existence and uniqueness of the solution of ODE

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x, y(x)) \\ y(x_0) = y_0 \end{cases} \quad \textbf{Initial Value Problem, IVP} \tag{3.7}$$

■ **Example 3.7** Consider the IVP

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2 y^{1/5} \\ y(x_0) = c > 0 \end{cases} \implies y = \left(\frac{4x^3}{15} + c^{4/5}\right)^{5/4}$$

which can be solved by the separation of variables:

$$c > 0 \implies y = \left(\frac{4x^3}{15} + c^{4/5}\right)^{5/4}.$$

However, when $c = 0$, the ODE does not have a unique solution. One can verify that y_1, y_2 given below are both solutions of this ODE:

$$y_1 = (\frac{4x^3}{15})^{5/4}, \qquad y_2 = 0$$

This example shows that even when f is very nice, the IVP may not have unique solution. The Picard-Lindelof theorem will give a clean condition on f ensuring the unique solvability of the IVP (3.7). ■