2.2. Monday for MAT3006

Reviewing.

1. Equivalent Metric:
$$d_1(\boldsymbol{x},\boldsymbol{y}) \le K d_2(\boldsymbol{x},\boldsymbol{y}) \le K' d_1(\boldsymbol{x},\boldsymbol{y})$$
In $\mathcal{C}[0,1]$, the metric d_1 and d_∞ are not equivalent:
For $f_n(x) = x^n n^2(1-x)$, $d_1(f_n,0) \to 1$ and $d_\infty(f_n,0) \to \infty$. Suppose on contrary that
$$d_1(f_n,0) \le K d_\infty(\boldsymbol{x},\boldsymbol{y}) \le K' d_1(\boldsymbol{x},\boldsymbol{y}).$$
Taking limit both sides, we imply the immediate term goes to infinite, which is a contradiction.
2. Continuous functions: the function f is continuous is equivalent to say for $\forall x_n \to x$, we have $f(x_n) \to f(x)$.
3. Open sets: Let (X,d) be a metric space. A set $U \subseteq X$ is open if for each $x \in U$, there exists $\rho_x > 0$ such that $B_{\rho_x}(x) \subseteq U$.

R Unless stated otherwise, we assume that
$$\mathcal{C}[a,b] \longleftrightarrow (\mathcal{C}[a,b], d_\infty)$$
$$\mathbb{R}^n \longleftrightarrow (\mathbb{R}^n, d_2)$$

2.2.1. Remark on Open and Closed Set

■ **Example 2.6** Let $X = \mathcal{C}[a,b]$, show that the set
$$U := \{f \in X \mid f(x) > 0, \forall x \in [a,b]\} \quad \text{is open.}$$

Take a point $f \in U$, then

$$\inf_{[a,b]} f(x) = m > 0.$$

Consider the ball $B_{m/2}(f)$, and for $\forall g \in B_{m/2}(f)$,

$$\begin{aligned}
|g(x)| &\geq |f(x)| - |f(x) - g(x)| \\
&\geq \inf_{[a,b]} |f(x)| - \sup_{[a,b]} |f(x) - g(x)| \\
&\geq m - \frac{m}{2} \\
&= \frac{m}{2} > 0, \ \forall x \in [a,b]
\end{aligned}$$

Therefore, we imply $g \in U$, i.e., $B_{m/2}(f) \subseteq U$, i.e., U is open in X. ■

Proposition 2.2 Let (X,d) be a metric space. Then

1. $\emptyset, X$ are open in X
2. If $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ are open in X, then $\bigcup_{\alpha \in \mathcal{A}}$ is also open in X
3. If $U_1, \dots, U_n$ are open in X, then $\bigcap_{i=1}^n U_i$ are open in X

R Note that $\bigcap_{i=1}^\infty U_i$ is not necessarily open if all U_i's are all open:

$$\bigcap_{i=1}^{\infty} \left(-\frac{1}{i}, 1 + \frac{1}{i}\right) = [0,1]$$

Definition 2.2 [Closed] The closed set in metric space (X,d) are the complement of open sets in X, i.e., any closed set in X is of the form $V = X \setminus U$, where U is oepn. ■

For example, in $\mathbb{R}$,

$$[a,b] = \mathbb{R} \setminus \{(-\infty, a) \bigcup (b, \infty)\}$$

Proposition 2.3

1. $\emptyset, X$ are closed in X
2. If $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ are closed subsets in X, then $\bigcap_{\alpha \in \mathcal{A}} V_\alpha$ is also closed in X
3. If $V_1, \dots, V_n$ are closed in X, then $\bigcup_{i=1}^n V_i$ is also closed in X.

R Whenever you say U is open or V is closed, you need to specify the underlying space, e.g.,

$$\textbf{Wrong}: U \text{ is open}$$

$$\textbf{Right}: U \text{ is open in } X$$

Proposition 2.4 The following two statements are equivalent:

1. The set V is closed in metric space (X,d).
2. If the sequence $\{v_n\}$ in V converges to x, then $x \in V$

Proof. *Necessity.*

Suppose on the contrary that $\{v_n\} \to x \notin V$. Since $X \setminus V \ni x$ is open, there exists an open ball $B_\varepsilon(x) \subseteq X \setminus V$.

Due to the convergence of sequence, there exists N such that $d(v_n,x) < \varepsilon$ for $\forall n \geq N$, i.e., $v_n \in B_\varepsilon(x)$, i.e., $v_n \notin V$, which contradicts to $\{v_n\} \subseteq V$.

Sufficiency.

Suppose on the contrary that V is not closed in X, i.e., $X \setminus V$ is not open, i.e., there exists $x \notin V$ such that for all open $U \ni x$, $U \bigcap V \neq \emptyset$. In particular, take

$$U_n = B_{1/n}(x), \implies \exists v_n \in B_{1/n}(x) \bigcap V,$$

i.e., $\{v_n\} \to x$ but $x \notin V$, which is a contradiction. ■

Proposition 2.5 Given two metric space (X,d) and (Y,ρ), the following statements are equivalent:

1. A function $f : (X,d) \to (Y,\rho)$ is continuous on X
2. For $\forall U \subseteq Y$ open in Y, $f^{-1}(U)$ is open in X.
3. For $\forall V \subseteq Y$ closed in Y, $f^{-1}(V)$ is closed in X.

■ **Example 2.7** The mapping $\Psi : \mathcal{C}[a,b] \to \mathbb{R}$ is defined as:

$$f \mapsto f(c)$$

where Ψ is called a **functional**.

Show that Ψ is continuous by using d_∞ metric on $\mathcal{C}[a,b]$:

1. Any open set in $\mathbb{R}$ can be written as countably union of open disjoint intervals, and therefore suffices to consider the pre-image $\Psi^{-1}(a,b) = \{f \mid f(c) \in (a,b)\}$. Following the similar idea in Example (2.6), it is clear that $\Psi^{-1}(a,b)$ is open in $(\mathcal{C}[a,b], d_\infty)$. Therefore, Ψ is continuous.
2. Another way is to apply definition.

We now study open sets in a subspace $(Y, d_Y) \subseteq (X, d_X)$, i.e.,

$$d_Y(y_1, y_2) := d_X(y_1, y_2).$$

Therefore, the open ball is defined as

$$\begin{aligned}
B_\varepsilon^Y(y) &= \{y' \in Y \mid d_Y(y, y') < \varepsilon\} \\
&= \{y' \in Y \mid d_X(y, y') < \varepsilon\} \\
&= \{y' \in X \mid d_X(y, y') < \varepsilon, y' \in Y\} \\
&= B_\varepsilon^X(y) \bigcap Y
\end{aligned}$$

Proposition 2.6 All open sets in the subspace $(Y, d_Y) \subseteq (X, d_X)$ are of the form $U \bigcap Y$, where U is open in X.

Corollary 2.1 For the subspace $(Y, d_Y) \subseteq (X, d_X)$, the mapping $i : (Y, d_Y) \to (X, d_X)$ with $i(y) = y, \forall y \in Y$ is continuous.

Proof. $i^{-1}(U) = U \bigcap Y$ for any subset $U \subseteq X$. The results follows from proposition (2.5). ■

(R) It's important to specify the underlying space to describe an open set.

For example, the interval $[0, \frac{1}{2})$ is not open in $\mathbb{R}$, while $[0, \frac{1}{2})$ is open in $[0,1]$, since

$$[0,\frac{1}{2}) = (-\frac{1}{2},\frac{1}{2})\bigcap[0,1].$$

2.2.2. Boundary, Closure, and Interior

Definition 2.3 Let (X,d) be a metric space, then

1. A point x is a **boundary point** of $S \subseteq X$ (denoted as $x \in \partial S$) if for any open $U \ni x$, then both $U \bigcap S, U \setminus S$ are non-empty.
   (one can replace U by $B_{1/n}(x)$, with $n = 1,2,\dots$)
2. The **closure** of S is defined as $\overline{S} = S \bigcup \partial S$.
3. A point x is an **interior point** of S (denoted as $x \in S^\circ$) if there $\exists U \ni x$ open such that $U \subseteq S$. We use S° to denote the set of interior points.

Proposition 2.7
1. The closure of S can be equivalently defined as

$$\overline{S} = \bigcap\{C \in X \mid C \text{ is closed and } C \supseteq S\}$$

   Therefore, $\overline{S}$ is the smallest closed set containing S.
2. The interior set of S can be equivalently defined as

$$S^\circ = \bigcup\{U \subseteq X \mid U \text{ is open and } U \subseteq S\}$$

   Therefore, S° is the largest open set contained in S.

Example 2.8 For $S = [0,\frac{1}{2}] \subseteq X$, we have

1. $\partial S = \{0,\frac{1}{2}\}$
2. $\overline{S} = [0,\frac{1}{2}]$
3. $S^\circ = (0,\frac{1}{2})$

Proof. 1. (a) Firstly, we show that $\overline{S}$ is closed, i.e., $X \setminus \overline{S}$ is open.

- Take $x \notin \overline{S}$. Since $x \notin \partial S$, there $\exists B_r(x) \ni x$ such that

$$B_r(x) \bigcap S, \quad \text{or} \quad B_r(x) \setminus S \text{ is } \emptyset.$$

- Since $x \notin S$, the set $B_r(x) \setminus S$ is not empty. Therefore, $B_r(x) \bigcap S = \emptyset$.
- It's clear that $B_{r/2}(x) \bigcap S = \emptyset$. We claim that $B_{r/2}(x) \bigcap \overline{S}$ is empty. Suppose on the contrary that

$$y \in B_{r/2}(x) \bigcap \partial S,$$

which implies that $B_{r/2}(y) \bigcap S \neq \emptyset$. Therefore,

$$B_{r/2}(y) \subseteq B_r(x) \implies B_r(x) \bigcap S \supseteq B_{r/2}(y) \bigcap S \neq \emptyset,$$

which is a contradiction.

Therefore, $x \in X \setminus \overline{S}$ implies $B_{r/2}(x) \bigcap \overline{S} = \emptyset$, i.e., $X \setminus \overline{S}$ is open, i.e., $\overline{S}$ is closed.

(b) Secondly, we show that $\overline{S} \subseteq C$, for any closed $C \supseteq S$, i.e., suffices to show $\partial S \subseteq C$.

Take $x \in \partial S$, and construct a sequence

$$x_n \in B_{1/n}(x) \bigcap S.$$

Here $\{x_n\}$ is a sequence in $S \subseteq C$ converging to x, which implies $x \in C$, due to the closeness of C in X.

Combining (a) and (b), the result follows naturally. (Question: do we need to show the well-defineness?)

2. Exercise. Show that

$$S^\circ = S \setminus \partial S = X \setminus (\overline{X \setminus S}).$$

Then it's clear that S° is open, and contained in S.

$\blacksquare$

The next lecture we will talk about compactness and sequential compactness.