1.5. Wednesday for MAT3006

Reviewing.

- Normed Space: a norm on a vector space
- Metric Space
- Open Ball

1.5.1. Convergence of Sequences

Since $\mathbb{R}^n$ and $\mathcal{C}[a,b]$ are both metric spaces, we can study the convergence in $\mathbb{R}^n$ and the functions defined on $[a,b]$ at the same time.

Definition 1.14 [Convergence] Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is **convergent** to x if $\forall\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n,x) < \varepsilon, \forall n \geq N.$$

We can denote the convergence by

$$x_n \to x, \quad \text{or} \quad \lim_{n\to\infty} x_n = x, \quad \text{or} \quad \lim_{n\to\infty} d(x_n,x) = 0$$

Proposition 1.10 If the limit of $\{x_n\}$ exists, then it is unique.

(R) Note that the proposition above does not necessarily hold for topology spaces.

Proof. Suppose $x_n \to x$ and $x_n \to y$, which implies

$$0 \leq d(x,y) \leq d(x,x_n) + d(x_n,y), \forall n$$

Taking the limit $n \to \infty$ both sides, we imply $d(x,y) = 0$, i.e., $x = y$. ■

■ **Example 1.16** 1. Consider the metric space $(\mathbb{R}^k, d_\infty)$ and study the convergence

$$\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x} \Longleftrightarrow \lim_{n\to\infty}\left(\max_{i=1\dots,k} |x_{n_i} - x_i|\right) = 0$$
$$\Longleftrightarrow \lim_{n\to\infty} |x_{n_i} - x_i| = 0, \forall i = 1,\dots,k$$
$$\Longleftrightarrow \lim_{n\to\infty} x_{n_i} = x_i,$$

i.e., the convergence defined in d_∞ is the same as the convergence defined in d_2.

2. Consider the convergence in the metric space $(\mathcal{C}[a,b], d_\infty)$:

$$\lim_{n\to\infty} f_n = f \Longleftrightarrow \lim_{n\to\infty}\left(\max_{[a,b]} |f_n(x) - f(x)|\right) = 0$$
$$\Longleftrightarrow \forall \varepsilon > 0, \forall x \in [a,b], \exists N_\varepsilon \text{ such that } |f_n(x) - f(x)| < \varepsilon, \forall n \geq N_\varepsilon$$

which is equivalent to the uniform convergence of functions, i.e., the convergence defined in d_2.

■

Definition 1.15 [Equivalent metrics] Let d and ρ be metrics on X.

1. We say ρ is **stronger** than d (or d is **weaker** than ρ) if

$$\exists K > 0 \text{ such that } d(x,y) \leq K\rho(x,y), \forall x,y \in X$$

2. The metrics d and ρ are equivalent if there exists $K_1, K_2 > 0$ such that

$$d(x,y) \leq K_1\rho(x,y) \leq K_2 d(x,y)$$

■

R The strongerness of ρ than d is depicted in the graph below:

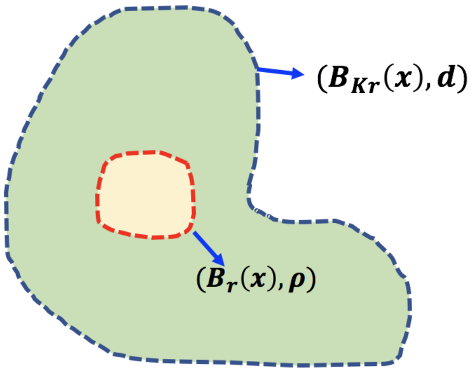

Figure 1.4: The open ball $(B_r(x), \rho)$ is contained by the open ball $(B_{Kr}(x), d)$

For each $x \in X$, consider the open ball $(B_r(x), \rho)$ and the open ball $(B_{Kr}(x), d)$:

$$B_r(x) = \{y \mid \rho(x,y) < r\}, \quad B_{Kr}(x) = \{z \mid d(x,z) < Kr\}.$$

For $y \in (B_r(x), \rho)$, we have $d(x,y) < K\rho(x,y) < Kr$, which implies $y \in (B_{Kr}(x), d)$, i.e, $(B_r(x), \rho) \subseteq (B_{Kr}(x), d)$ for any $x \in X$ and $r > 0$.

■ **Example 1.17** 1. d_1, d_2, d_∞ in $\mathbb{R}^n$ are equivalent

$$d_1(\boldsymbol{x}, \boldsymbol{y}) \le d_\infty(\boldsymbol{x}, \boldsymbol{y}) \le n d_1(\boldsymbol{x}, \boldsymbol{y})$$
$$d_2(\boldsymbol{x}, \boldsymbol{y}) \le d_\infty(\boldsymbol{x}, \boldsymbol{y}) \le \sqrt{n} d_2(\boldsymbol{x}, \boldsymbol{y})$$

We use two relation depiected in the figure below to explain these two inequalities:

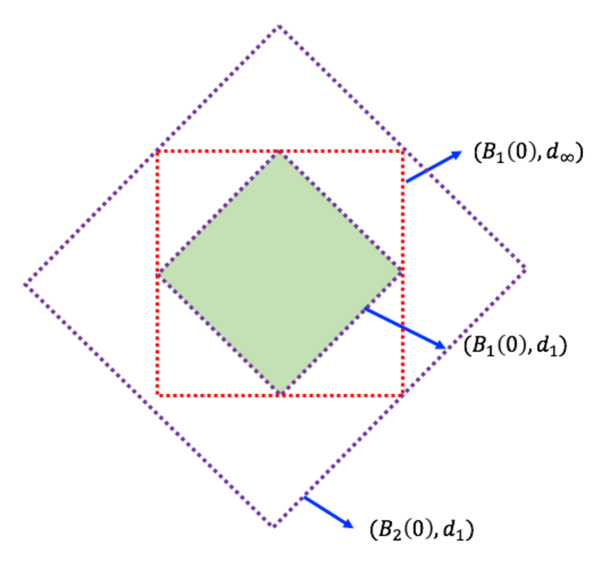

Figure 1.5: The diagram for the relation $(B_1(x), d_1) \subseteq (B_\infty(x), d_\infty) \subseteq (B_2(x), d_1)$ on $\mathbb{R}^2$

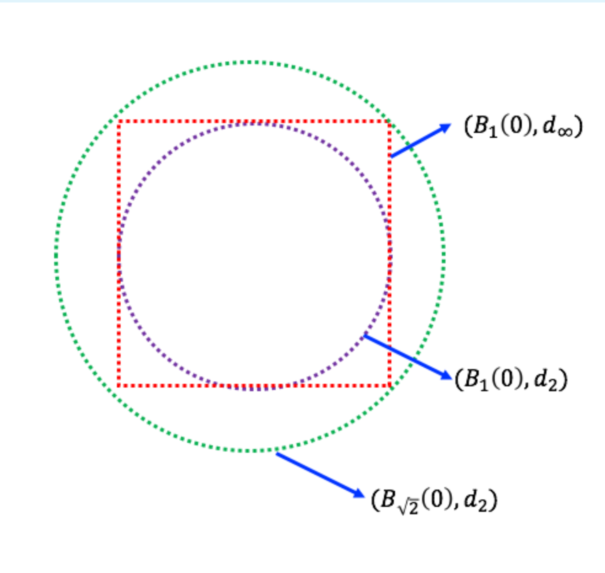

Figure 1.6: The diagram for the relation $(B_1(x), d_2) \subseteq (B_\infty(x), d_\infty) \subseteq (B_{\sqrt{2}}(x), d_2)$ on $\mathbb{R}^2$

It's easy to conclude the simple generalization for example (1.16):

Proposition 1.11 If d and ρ are equivalent, then

$$\lim_{n\to\infty} d(x_n, x) = 0 \Longleftrightarrow \lim_{n\to\infty} \rho(x_n, x) = 0$$

Note that this does not necessarily hold for topology spaces.

2. Consider d_1, d_∞ in $\mathcal{C}[a,b]$:

$$d_1(f,g) := \int_a^b |f-g|\,\mathrm{d}x \le \int_a^b \sup_{[a,b]} |f-g|\,\mathrm{d}x = (b-a)d_\infty(f,g),$$

i.e., d_∞ is stronger than d_1. Question: Are they equivalent? **No**.

Justification. Consider $f_n(x) = n^2 x^n (1-x)$. Check that

$$\lim_{n\to\infty} d_1(f_n(x), 1) = 0, \quad \text{but } d_\infty(f_n(x), 1) \to \infty$$

The peak of f_n may go to infinite, while the integration converges to zero. Therefore d_1 and d_∞ have different limits. We will discuss this topic at Lebsegue integration again. ■

1.5.2. Continuity

Definition 1.16 [Continuity] Let $f : (X,d) \to (Y,d)$ be a function and $x_0 \in X$. Then f is continuous at x_0 if $\forall \varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x,x_0) < \delta \implies \rho(f(x), f(x_0)) < \varepsilon$$

The function f is continuous in X if f is continuous for all $x_0 \in X$.

Proposition 1.12 The function f is continuous at x if and only if for all $\{x_n\} \to x$ under d, $f(x_n) \to f(x)$ under ρ.

Proof. *Necessity:* Given $\varepsilon > 0$, by continuity,

$$d(x,x') < \delta \implies \rho(f(x'), f(x)) < \varepsilon. \tag{1.3}$$

Consider the sequence $\{x_n\} \to x$, then there exists N such that $d(x_n, x) < \delta$ for $\forall n \ge N$. By applying (1.3), $\rho(f(x_n), f(x)) < \varepsilon$ for $\forall n \ge N$, i.e., $f(x_n) \to f(x)$.

Sufficiency: Assume that f is not continuous at x, then there exists ε_0 such that for $\delta_n = \frac{1}{n}$, there exists x_n such that

$$d(x_n, x) < \delta_n, \text{ but } \rho(f(x_n), f(x)) > \varepsilon_0.$$

Then $\{x_n\} \to x$ by our construction, while $\{f(x_n)\}$ does not converge to $f(x)$, which is a contradiction. ■

Corollary 1.2 If the function $f : (X,d) \to (Y,\rho)$ is continuous at x, the function $g : (Y,\rho) \to (Z,m)$ is continuous at $f(x)$, then $g \circ f : (X,d) \to (Z,m)$ is continuous at x.

Proof. Note that

$$\{x_n\} \to x \overset{(a)}{\Longrightarrow} \{f(x_n)\} \to f(x) \overset{(b)}{\Longrightarrow} \{g(f(x_n))\} \to g(f(x)) \overset{(c)}{\Longrightarrow} g \circ f \text{ is continuous at } x.$$

where $(a),(b),(c)$ are all by proposition (1.12). ■

1.5.3. Open and Closed Sets

We have open/closed intervals in $\mathbb{R}$, and they are important in some theorems (e.g, continuous functions bring closed intervals to closed intervals).

Definition 1.17 [Open] Let (X,d) be a metric space. A set $U \subseteq X$ is open if for each $x \in U$, there exists $\rho_x > 0$ such that $B_{\rho_x}(x) \subseteq U$. The empty set $\emptyset$ is defined to be open.
■

■ **Example 1.18** Let $(\mathbb{R}, d_2 \text{ or } d_\infty)$ be a metric space. The set $U = (a,b)$ is open. ■

Proposition 1.13
1. Let (X,d) be a metric space. Then all open balls $B_r(x)$ are open
2. All open sets in X can be written as a union of open balls.

Proof. 1. Let $y \in B_r(x)$, i.e., $d(x,y) := q < r$. Consider the open ball $B_{(r-q)/2}(y)$. It suffices to show $B_{(r-q)/2}(y) \subseteq B_r(x)$. For any $z \in B_{(r-q)/2}(y)$,

$$d(x,z) \leq d(x,y) + d(y,z) < q + \frac{r-q}{2} = \frac{r+q}{2} < r.$$

The proof is complete.

2. Let $U \subseteq X$ be open, i.e., for $\forall x \in U$, there exists $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subseteq U$. Therefore

$$\{x\} \subseteq B_{\varepsilon_x}(x) \subseteq U, \forall x \in U$$

which implies

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B_{\varepsilon_x}(x) \subseteq U,$$

i.e., $U = \bigcup_{x \in U} B_{\varepsilon_x}(x)$.

■