15.2. Monday for MAT3006

15.2.1. Applications on the Tonell's and Fubini's Theorem

Theorem 15.2 — **Tonell.** Let $f : \mathbb{R}^2 \to [0, \infty]$ be a measurable function (i.e., $f^{-1}((a, \infty]) \in \mathcal{M} \otimes \mathcal{M}$), then

$$\int f \, d\pi = \int \left(\int f(x, y) \, dx \right) dy = \int \left(\int f(x, y) \, dy \right) dx$$

Theorem 15.3 — **Fubini.** Let $f : \mathbb{R}^2 \to [-\infty, \infty]$ be integrable (i.e., $f = f^+ - f^-$ with $f^\pm : \mathbb{R}^2 \to [0, \infty]$ measurable and $\int f^\pm \, dx < \infty$), then

$$\int f \, d\pi = \int \left(\int f(x, y) \, dx \right) dy = \int \left(\int f(x, y) \, dy \right) dx$$

Corollary 15.2 Suppose that $f : \mathbb{R}^2 \to [-\infty, \infty]$ is measurable, and either

$$\int \left(\int |f(x, y)| \, dx \right) dy \tag{15.1a}$$

or

$$\int \left(\int |f(x, y)| \, dy \right) dx \tag{15.1b}$$

exists, then f is integrable, and the result of Fubini follows. (i.e., one can switch the order of integration as long as the integral of $|f|$ exists)

Proof. If (15.1a) or (15.1b) exists (is finite), then Tonell's Theorem implies that $|f|$ is integrable, which implies f is integrable.

Therefore, the assumption of Fubini's theorem holds, and the proof is comptete. ■

(R) The advantage for corollary (15.2) is that computing (15.1a) or (15.1b) is easier than showing the integrability of f in general.

■ **Example 15.2** Compute the double integral

$$\int_0^1 \int_0^x \sqrt{\frac{1-y}{x-y}}\,\mathrm{d}y\,\mathrm{d}x.$$

Construct the function $f(x,y) := \sqrt{\frac{1-y}{x-y}}\mathcal{X}_E(x,y)$, with E shown in Fig. (15.1).

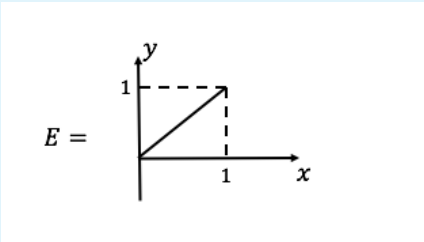

Figure 15.1: Illustration for integral domain E

We want to compute $\int f(x,y)\,\mathrm{d}\pi$ and show that

$$\int_0^1 \int_0^x \sqrt{\frac{1-y}{x-y}}\,\mathrm{d}y\,\mathrm{d}x = \int f(x,y)\,\mathrm{d}\pi.$$

- Consider the integral

$$\begin{aligned}
\int \left(\int f(x,y)\,\mathrm{d}x\right)\mathrm{d}y &= \int \left(\int f(x,y)\mathcal{X}_{E_y}\,\mathrm{d}x\right)\mathcal{X}_{E_x}\,\mathrm{d}y && \text{(15.2a)}\\
&= \int_0^1 \left(\int_y^1 \sqrt{\frac{1-y}{x-y}}\,\mathrm{d}x\right)\mathrm{d}y && \text{(15.2b)}\\
&= \int_0^1 \sqrt{1-y}\left(\int_y^1 \frac{1}{\sqrt{x-y}}\,\mathrm{d}x\right)\mathrm{d}y && \text{(15.2c)}\\
&= \int_0^1 \sqrt{1-y}\left(\int_0^{1-y} \frac{1}{\sqrt{t}}\,\mathrm{d}t\right)\mathrm{d}y && \text{(15.2d)}\\
&= \int_0^1 \sqrt{1-y}\cdot(2\sqrt{1-y})\,\mathrm{d}y && \text{(15.2e)}\\
&= 2\int_0^1 (1-y)\,\mathrm{d}y && \text{(15.2f)}\\
&= 1 && \text{(15.2g)}
\end{aligned}$$

- Therefore, $\int(\int |f(x,y)|\,\mathrm{d}x)\,\mathrm{d}y < \infty$. Mreover, f is continous on E°, i.e., measurable on E° (it's clear that a continous function is measurable). Since ∂E is null, we imply f is measurable on $E := E^\circ \cup \partial E$.

- Therefore, the assumption of Corollary (15.2) holds, and we imply that

$$\int \left(\int f(x,y) \, \mathrm{d}y \right) \mathrm{d}x = \int \left(\int f(x,y) \, \mathrm{d}x \right) \mathrm{d}y$$

It's clear that

$$\int_0^1 \int_0^x \sqrt{\frac{1-y}{x-y}} \, \mathrm{d}y \, \mathrm{d}x = \int \left(\int f(x,y) \, \mathrm{d}y \right) \mathrm{d}x,$$

and therefore

$$\int_0^1 \int_0^x \sqrt{\frac{1-y}{x-y}} \, \mathrm{d}y \, \mathrm{d}x = 1.$$

Process of Completion. We have two measures on $\mathbb{R}^2$:

- $\mathcal{M} \otimes \mathcal{M}$, and
- $\mathcal{M}_{\mathbb{R}^2}$, given by

$$\mathcal{M}_{\mathbb{R}^2} = \{E \subseteq \mathbb{R}^2 \mid m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \text{ for all subsets } A \subseteq \mathbb{R}^2\}$$

Here $\mathcal{M}_{\mathbb{R}^2}$ equals the completion of $\mathcal{M} \otimes \mathcal{M}$, i.e., all $E \subseteq \mathcal{M}_{\mathbb{R}^2}$ can be decomposed as

$$E = B \cup (E \setminus B),$$

where $B \in \mathcal{M} \otimes \mathcal{M}$ and $E \setminus B \in \mathcal{M}_{\mathbb{R}^2}$ with $\pi(E \setminus B) = 0$.

Question: does Tonell's theorem holds for (Lebesgue) measurable functions $f : \mathbb{R}^2 \to [0, \infty]$ (i.e., $f^{-1}((a, \infty]) \in \mathcal{M}_{\mathbb{R}^2}$ for any $a \in [0, \infty)$?)

Answer: Yes. To see so, we just need the following proposition

Proposition 15.4 Let $(\mathbb{R}^2, \mathcal{M}_{\mathbb{R}^2}, \pi)$ be the Lebesgue measure on $\mathbb{R}^2$, and $N \in \mathcal{M}_{\mathbb{R}^2}$ be such that $\pi(N) = 0$. Then for almost all values of $x \in \mathbb{R}$, $N_x \in \mathcal{M}$ and $m_Y(N_x) = 0$.

Proof. For $N \in \mathcal{M}_{\mathbb{R}^2}$. By hw3, there exists $B' \in \mathcal{M} \otimes \mathcal{M}$ such that $N \subseteq B'$, with

$$\pi(B') = \pi(N).$$

If N is null, then $\pi(B') = 0$. By Tonell's theorem on $M \otimes \mathcal{M}$, we imply

$$\pi(B') = \int m_Y(B'_x)\,\mathrm{d}x = \int m_X(B'_y)\,\mathrm{d}y = 0$$

Therefore, $m_Y(B'_x) = 0$ for almost all $x \in \mathbb{R}$. Since $N \subseteq B'$, we imply $N_x \subseteq B'_x$, i.e., N_x is also a null set. Therefore, $N_x \in \mathcal{M}$ and $m_Y(N_x) = 0$. ■

▪ **Example 15.3** Consider the integral

$$\int_0^\infty \int_0^\infty y e^{-y^2(1+x^2)}\,\mathrm{d}y\,\mathrm{d}x$$

Define $f(x,y) = ye^{-y^2(1+x^2)}$, which is continous on $(0,\infty)\times(0,\infty)$, and therefore measurable. It follows that

$$\begin{aligned}
\int_0^\infty \int_0^\infty f(x,y)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^\infty \left(\lim_{n\to\infty}\int_0^n f(x,y)\,\mathrm{d}y\right)\mathrm{d}x && (15.3a)\\
&= \int_0^\infty \left(\frac{1}{1+x^2}\frac{1}{2}\right)\mathrm{d}x && (15.3b)\\
&= \lim_{n\to\infty}\int_0^n \frac{1}{2}\frac{1}{1+x^2}\,\mathrm{d}x && (15.3c)\\
&= \frac{\pi}{4} && (15.3d)
\end{aligned}$$

where (15.3a) is by applying MCT I on the function $f(x,y)\mathcal{X}_{[0,n]}$; (15.3b) and (15.3d) is by computation; (15.3c) is by applying MCT I on the function $\frac{1}{1+x^2}\frac{1}{2}\mathcal{X}_{[0,n]}$.

By corollary (15.2),

$$\int_0^\infty \int_0^\infty y e^{-y^2(1+x^2)}\,\mathrm{d}x\,\mathrm{d}y = \frac{\pi}{4}$$

Or equivalently,

$$\int_0^\infty y e^{-y^2} \int_0^\infty e^{-x^2y^2}\,\mathrm{d}x\,\mathrm{d}y = \frac{\pi}{4}$$

By applying MCT I on $e^{-x^2y^2}\mathcal{X}_{[0,n]}$, we have

$$\int_0^\infty ye^{-y^2} \lim_{n\to\infty} \int_0^n e^{-x^2y^2}\, dx\, dy = \frac{\pi}{4}$$

By change of variable with $t = xy$, we imply

$$\int_0^\infty ye^{-y^2} \lim_{n\to\infty} \frac{1}{y} \int_0^{ny} e^{-t^2}\, dt\, dy = \frac{\pi}{4}$$

Or equivalently,

$$\int_0^\infty e^{-y^2} \int_0^\infty e^{-t^2}\, dt\, dy = \frac{\pi}{4}$$

Therefore, we conclude that

$$\left(\int_0^\infty e^{-y^2}\, dy\right)^2 = \frac{\pi}{4} \implies \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

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